Dušan Djukić Vladimir Janković
Ivan Matić Nikola Petrović

IMO Shortlist 2005

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Problems

1.1 The Forty-Sixth IMO
Mérida, Mexico, July 8–19, 2005

1.1.1 Contest Problems

First Day (July 13)

1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

3. Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

Second Day (July 14)

4. Consider the sequence $a_1, a_2, \ldots$ defined by
$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots).$$
Determine all positive integers that are relatively prime to every term of the sequence.

5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF

meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

1.1.2 Shortlisted Problems

1. **A1 (ROM)** Find all monic polynomials $p(x)$ with integer coefficients of degree two for which there exists a polynomial $q(x)$ with integer coefficients such that $p(x)q(x)$ is a polynomial having all coefficients ± 1.
2. **A2 (BUL)** Let $\mathbb{R}^+$ denote the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)f(y) = 2f(x+yf(x))$$
for all positive real numbers x and y.
3. **A3 (CZE)** Four real numbers p,q,r,s satisfy
$$p+q+r+s=9 \quad \text{and} \quad p^2+q^2+r^2+s^2=21.$$
Prove that $ab-cd \geq 2$ holds for some permutation (a,b,c,d) of (p,q,r,s).
4. **A4 (IND)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation
$$f(x+y)+f(x)f(y) = f(xy)+2xy+1$$
for all real x and y.
5. **A5 (KOR)**$^{\text{IMO3}}$ Let x,y and z be positive real numbers such that $xyz \geq 1$. Prove that
$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$
6. **C1 (AUS)** A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.
7. **C2 (IRN)** Let k be a fixed positive integer. A company has a special method to sell sombreros. Each customer can convince two persons to buy a sombrero after he/she buys one; convincing someone already convinced does not count. Each

of these new customers can convince two others and so on. If each one of the two customers convinced by someone makes at least k persons buy sombreros (directly or indirectly), then that someone wins a free instructional video. Prove that if n persons bought sombreros, then at most $n/(k+2)$ of them got videos.

8. **C3 (IRN)** In an $m \times n$ rectangular board of mn unit squares, *adjacent* squares are ones with a common edge, and a *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be colored black or white. Let N denote the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge, and let M denote the number of colorings in which there exist at least two non-intersecting black paths from the left edge to the right edge. Prove that $N^2 \geq 2^{mn}M$.

9. **C4 (COL)** Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1 \dots P_n$ with a positive integer less than or equal to r so that:
   (i) every integer between 1 and r occurs as a label;
   (ii) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.
   Given these conditions:
   (a) Determine the largest positive integer r for which this can be done.
   (b) For that value of r, how many such labellings are there?

10. **C5 (SMN)** There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n-1$ is not divisible by 3.

11. **C6 (ROM)**$^{\text{IMO6}}$ In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $2/5$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

12. **C7 (USA)** Let $n \geq 1$ be a given integer, and let $a_1, \dots, a_n$ be a sequence of integers such that n divides the sum $a_1 + \cdots + a_n$. Show that there exist permutations σ and τ of $1, 2, \dots, n$ such that $\sigma(i) + \tau(i) \equiv a_i \pmod{n}$ for all $i = 1, \dots, n$.

13. **C8 (BUL)** Let M be a convex n-gon, $n \geq 4$. Some $n-3$ of its diagonals are colored green and some other $n-3$ diagonals are colored red, so that no two diagonals of the same color meet inside M. Find the maximum possible number of intersection points of green and red diagonals inside M.

14. **G1 (GRE)** In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has center I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

15. **G2 (ROM)**IMO1 Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC; B_1, B_2 on CA; C_1, C_2 on AB. These points are vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

16. **G3 (UKR)** Let $ABCD$ be a parallelogram. A variable line l passing through the point A intersects the rays BC and DC at points X and Y, respectively. Let K and L be the centers of the excircles of triangles ABX and ADY, touching the sides BX and DY, respectively. Prove that the size of angle KCL does not depend on the choice of the line l.

17. **G4 (POL)**IMO5 Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

18. **G5 (ROM)** Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocenter and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

19. **G6 (RUS)** The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

20. **G7 (KOR)** In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that $p(ABC)p(PQR) \geq p(DEF)^2$, where $p(T)$ denotes the perimeter of triangle T.

21. **N1 (POL)**IMO4 Consider the sequence $a_1, a_2, \dots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \dots).$$

Determine all positive integers that are relatively prime to every term of the sequence.

22. **N2 (NET)**IMO2 Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \dots, a_n$ leave n different remainders on division by n. Prove that each integer occurs exactly once in the sequence.

23. **N3 (MON)** Let a, b, c, d, e and f be positive integers. Suppose that the sum $S = a + b + c + d + e + f$ divides both $abc + def$ and $ab + bc + ca - de - ef - fd$. Prove that S is composite.

24. **N4 (COL)** Find all positive integers $n > 1$ for which there exists a unique integer a with $0 < a \leq n!$ such that $a^n + 1$ is divisible by $n!$.

25. **N5 (NET)** Denote by $d(n)$ the number of divisors of the positive integer n. A positive integer n is called *highly divisible* if $d(n) > d(m)$ for all positive integers $m < n$. Two highly divisible integers m and n with $m < n$ are called consecutive if there exists no highly divisible integer s satisfying $m < s < n$.
    (a) Show that there are only finitely many pairs of consecutive highly divisible integers of the form (a, b) with $a|b$.
    (b) Show that for every prime number p there exist infinitely many positive highly divisible integers r such that pr is also highly divisible.

26. **N6 (IRN)** Let a and b be positive integers such that $a^n + n$ divides $b^n + n$ for every positive integer n. Show that $a = b$.

27. **N7 (RUS)** Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.

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Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2005

1. Clearly, $p(x)$ has to be of the form $p(x) = x^2 + ax \pm 1$ where a is an integer. For $a = \pm 1$ and $a = 0$ polynomial p has the required property: it suffices to take $q = 1$ and $q = x + 1$, respectively.
   Suppose now that $|a| \geq 2$. Then $p(x)$ has two real roots, say x_1, x_2, which are also roots of $p(x)q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_i = \pm 1$. Thus
   $$1 = \left| \frac{a_{n-1}}{x_i} + \cdots + \frac{a_0}{x_i^n} \right| \leq \frac{1}{|x_i|} + \cdots + \frac{1}{|x_i|^n} < \frac{1}{|x_i| - 1}$$
   which implies $|x_1|, |x_2| < 2$. This immediately rules out the case $|a| \geq 3$ and the polynomials $p(x) = x^2 \pm 2x - 1$. The remaining two polynomials $x^2 \pm 2x + 1$ satisfy the condition for $q(x) = x \mp 1$.
   Summing all, the polynomials $p(x)$ with the desired property are $x^2 \pm x \pm 1$, $x^2 \pm 1$ and $x^2 \pm 2x + 1$.

2. Given $y > 0$, consider the function $\varphi(x) = x + yf(x)$, $x > 0$. This function is injective: indeed, if $\varphi(x_1) = \varphi(x_2)$ then $f(x_1)f(y) = f(\varphi(x_1)) = f(\varphi(x_2)) = f(x_2)f(y)$, so $f(x_1) = f(x_2)$, so $x_1 = x_2$ by the definition of φ. Now if $x_1 > x_2$ and $f(x_1) < f(x_2)$, we have $\varphi(x_1) = \varphi(x_2)$ for $y = \frac{x_1 - x_2}{f(x_2) - f(x_1)} > 0$, which is impossible; hence f is non-decreasing. The functional equation now yields $f(x)f(y) = 2f(x + yf(x)) \geq 2f(x)$ and consequently $f(y) \geq 2$ for $y > 0$. Therefore
   $$f(x + yf(x)) = f(xy) = f(y + xf(y)) \geq f(2x)$$
   holds for arbitrarily small $y > 0$, implying that f is constant on the interval $(x, 2x]$ for each $x > 0$. But then f is constant on the union of all intervals $(x, 2x]$ over all $x > 0$, that is, on all of $\mathbb{R}^+$. Now the functional equation gives us $f(x) = 2$ for all x, which is clearly a solution.

   Second Solution. In the same way as above we prove that f is non-decreasing, hence its discontinuity set is at most countable. We can extend f to $\mathbb{R} \cup \{0\}$ by defining $f(0) = \inf_x f(x) = \lim_{x \to 0} f(x)$ and the new function f is continuous at 0 as well. If x is a point of continuity of f we have $f(x)f(0) = \lim_{y \to 0} f(x)f(y) = \lim_{y \to 0} 2f(x + yf(x)) = 2f(x)$, hence $f(0) = 2$. Now, if f is continuous at $2y$ then $2f(y) = \lim_{x \to 0} f(x)f(y) = \lim_{x \to 0} 2f(x + yf(x)) = 2f(2y)$. Thus $f(y) = f(2y)$, for all but countably many values of y. Being non-decreasing f is a constant, hence $f(x) = 2$.

3. Assume w.l.o.g. that $p \geq q \geq r \geq s$. We have
   $$(pq + rs) + (pr + qs) + (ps + qr) = \frac{(p + q + r + s)^2 - p^2 - q^2 - r^2 - s^2}{2} = 30.$$
   It is easy to see that $pq + rs \geq pr + qs \geq ps + qr$ which gives us $pq + rs \geq 10$. Now setting $p + q = x$ we obtain $x^2 + (9 - x)^2 = (p + q)^2 + (r + s)^2 = 21 + 2(pq + rs) \geq 41$ which is equivalent to $(x - 4)(x - 5) \geq 0$. Since $x = p + q \geq r + s$ we conclude that $x \geq 5$. Thus

$$25 \le p^2+q^2+2pq = 21-(r^2+s^2)+2pq \le 21+2(pq-rs),$$

or $pq-rs \ge 2$, as desired.
Remark. The quadruple $(p,q,r,s)=(3,2,2,2)$ shows that the estimate 2 is the best possible.

4. Setting $y=0$ yields $(f(0)+1)(f(x)-1)=0$, and since $f(x)=1$ for all x is impossible, we get $f(0)=-1$. Now plugging in $x=1$ and $y=-1$ gives us $f(1)=1$ or $f(-1)=0$. In the first case setting $x=1$ in the functional equation yields $f(y+1)=2y+1$, i.e. $f(x)=2x-1$ which is one solution.
Suppose now that $f(1)=a\neq 1$ and $f(-1)=0$. Plugging $(x,y)=(z,1)$ and $(x,y)=(-z,-1)$ in the functional equation yields

$$\begin{aligned} f(z+1) &= (1-a)f(z)+2z+1 \\ f(-z-1) &= f(z)+2z+1. \end{aligned} \qquad (*)$$

It follows that $f(z+1)=(1-a)f(-z-1)+a(2z+1)$, i.e. $f(x)=(1-a)f(-x)+a(2x-1)$. Analogously $f(-x)=(1-a)f(x)+a(-2x-1)$, which together with the previous equation yields

$$(a^2-2a)f(x) = -2a^2x-(a^2-2a).$$

Now $a=2$ is clearly impossible. For $a\notin\{0,2\}$ we get $f(x)=\frac{-2ax}{a-2}-1$. This function satisfies the requirements only for $a=-2$, giving the solution $f(x)=-x-1$. In the remaining case, when $a=0$, we have $f(x)=f(-x)$. Setting $y=z$ and $y=-z$ in the functional equation and subtracting yields $f(2z)=4z^2-1$, so $f(x)=x^2-1$ which satisfies the equation.
Thus the solutions are $f(x)=2x-1$, $f(x)=-x-1$ and $f(x)=x^2-1$.

5. The desired inequality is equivalent to

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \le 3. \qquad (*)$$

By the Cauchy inequality we have $(x^5+y^2+z^2)(yz+y^2+z^2) \ge (x^{5/2}(yz)^{1/2}+y^2+z^2)^2 \ge (x^2+y^2+z^2)^2$ and therefore

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \le \frac{yz+y^2+z^2}{x^2+y^2+z^2}.$$

We get analogous inequalities for the other two summands in $(*)$. Summing these up yields

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2} \le 2+\frac{xy+yz+zx}{x^2+y^2+z^2},$$

which together with the well-known inequality $x^2+y^2+z^2 \ge xy+yz+zx$ gives us the result.

Second solution. Multiplying the both sides with the common denominator and using the notation as in Chapter 2 (*Muirhead's inequality*) we get

$$T_{5,5,5}+4T_{7,5,0}+T_{5,2,2}+T_{9,0,0} \geq T_{5,5,2}+T_{6,0,0}+2T_{5,4,0}+2T_{4,2,0}+T_{2,2,2}.$$

By Schur's and Muirhead's inequalities we have that $T_{9,0,0}+T_{5,2,2} \geq 2T_{7,2,0} \geq 2T_{7,1,1}$. Since $xyz \geq 1$ we have that $T_{7,1,1} \geq T_{6,0,0}$. Therefore

$$T_{9,0,0}+T_{5,2,2} \geq 2T_{6,0,0} \geq T_{6,0,0}+T_{4,2,0}. \tag{1}$$

Moreover, Muirhead's inequality combined with $xyz \geq 1$ gives us $T_{7,5,0} \geq T_{5,5,2}$, $2T_{7,5,0} \geq 2T_{6,5,1} \geq 2T_{5,4,0}$, $T_{7,5,0} \geq T_{6,4,2} \geq T_{4,2,0}$, and $T_{5,5,5} \geq T_{2,2,2}$. Adding these four inequalities to (1) yields the desired result.

6. A room will be called *economic* if some of its lamps are on and some are off. Two lamps sharing a switch will be called *twins*. The twin of a lamp l will be denoted $\overline{l}$.
Suppose we have arrived at a state with the minimum possible number of uneconomic rooms, and that this number is strictly positive. Let us choose any uneconomic room, say R_0, and a lamp l_0 in it. Let $\overline{l_0}$ be in a room R_1. Switching l_0 we make R_0 economic; thereby, since the number of uneconomic rooms cannot be decreased, this change must make room R_1 uneconomic. Now choose a lamp l_1 in R_1 having the twin $\overline{l_1}$ in a room R_2. Switching l_1 makes R_1 economic, and thus must make R_2 uneconomic. Continuing in this manner we obtain a sequence $l_0, l_1, \dots$ of lamps with l_i in a room R_i and $\overline{l_i} \neq l_{i+1}$ in R_{i+1} for all i. The lamps $l_0, l_1, \dots$ are switched in this order. This sequence has the property that switching l_i and $\overline{l_i}$ makes room R_i economic and room R_{i+1} uneconomic.
Let $R_m = R_k$ with $m > k$ be the first repetition in the sequence (R_i). Let us stop switching the lamps at l_{m-1}. The room R_k was uneconomic prior to switching l_k. Thereafter lamps l_k and $\overline{l_{m-1}}$ have been switched in R_k, but since these two lamps are distinct (indeed, their twins $\overline{l_k}$ and l_{m-1} are distinct), the room R_k is now economic as well as all the rooms $R_0, R_1, \dots, R_{m-1}$. This decreases the number of uneconomic rooms, contradicting our assumption.

7. Let v be the number of video winners. One easily finds that for $v = 1$ and $v = 2$, the number n of customers is at least $2k+3$ and $3k+5$ respectively. We prove by induction on v that if $n \geq k+1$ then $n \geq (k+2)(v+1)-1$.
We can assume w.l.o.g. that the total number n of customers is minimum possible for given $v > 0$. Consider a person P who was convinced by nobody but himself. Then P must have won a video; otherwise P could be removed from the group without decreasing the number of video winners. Let Q and R be the two persons convinced by P. We denote by $\mathscr{C}$ the set of persons made by P through Q to buy a sombrero, including Q, and by $\mathscr{D}$ the set of all other customers excluding P. Let x be the number of video winners in $\mathscr{C}$. Then there are $v-x-1$ video winners in $\mathscr{D}$. We have $|\mathscr{C}| \geq (k+2)(x+1)-1$, by induction hypothesis if $x > 0$ and because P is a winner if $x = 0$. Similarly, $|\mathscr{D}| \geq (k+2)(v-x)-1$. Thus $n \geq 1+(k+2)(x+1)-1+(k+2)(v-x)-1$, i.e. $n \geq (k+2)(v+1)-1$.

8. Suppose that a two-sided $m \times n$ board T is considered, where exactly k of the squares are transparent. A transparent square is colored only on one side (then it looks the same from the other side), while a non-transparent one needs to be colored on both sides, not necessarily in the same color.
Let $C = C(T)$ be the set of colorings of the board in which there exist two black paths from the left edge to the right edge, one on top and one underneath, not intersecting at any transparent square. If $k = 0$ then $|C| = N^2$. We prove by induction on k that $2^k|C| \le N^2$: this will imply the statement of the problem, as $|C| = M$ for $k = mn$.
Let q be a fixed transparent square. Consider any coloring B in C: If q is converted into a non-transparent square, a new board T' with $k-1$ transparent squares is obtained, so by the induction hypothesis $2^{k-1}|C(T')| \le N^2$. Since B contains two black paths at most one of which passes through q, coloring q in either color on the other side will result in a coloring in C'; hence $|C(T')| \ge 2|C(T)|$, implying $2^k|C(T)| \le N^2$ and finishing the induction.
Second solution. By *path* we shall mean a black path from the left edge to the right edge. Let $\mathscr{A}$ denote the set of pairs of $m \times n$ boards each of which has a path. Let $\mathscr{B}$ denote the set of pairs of boards such that the first board has two non-intersecting paths. Obviously, $|\mathscr{A}| = N^2$ and $|\mathscr{B}| = 2^{mn}M$. To show $|\mathscr{A}| \ge |\mathscr{B}|$ we will construct an injection $f : \mathscr{B} \to \mathscr{A}$.
Among paths on a given board we define path x to be *lower* than y if the set of squares "under" x is a subset of the squares under y. This relation is a relation of incomplete order. However, for each board with at least one path there exists the lowest path (comparing two intersecting paths, we can always take the "lower branch" on each non-intersecting segment). Now, for a given element of $\mathscr{B}$, we "swap" the lowest path and all squares underneath on the first board with the corresponding points on the other board. This swapping operation is the desired injection f. Indeed, since the first board still contains the highest path (which didn't intersect the lowest one), the new configuration belongs to $\mathscr{A}$. On the other hand, this configuration uniquely determines the lowest path on the original element of $\mathscr{B}$; hence no two different elements of $\mathscr{B}$ can go to the same element of $\mathscr{A}$. This completes the proof.

9. Let $[XY]$ denote the label of segment XY, where X and Y are vertices of the polygon. Consider any segment MN with the maximum label $[MN] = r$. By condition (ii), for any $P_i \ne M,N$, exactly one of P_iM and P_iN is labelled by r. Thus the set of all vertices of the n-gon splits into two complementary groups: $\mathscr{A} = \{P_i \mid [P_iM] = r\}$ and $\mathscr{B} = \{P_i \mid [P_iN] = r\}$. We claim that a segment XY is labelled by r if and only if it joins two points from different groups. Assume w.l.o.g. that $X \in \mathscr{A}$. If $Y \in \mathscr{A}$, then $[XM] = [YM] = r$, so $[XY] < r$. If $Y \in \mathscr{B}$, then $[XM] = r$ and $[YM] < r$, so $[XY] = r$ by (ii), as we claimed.
We conclude that a labelling satisfying (ii) is uniquely determined by groups $\mathscr{A}$ and $\mathscr{B}$ and labellings satisfying (ii) within A and B.
   (a) We prove by induction on n that the greatest possible value of r is $n-1$. The degenerate cases $n = 1,2$ are trivial. If $n \ge 3$, the number of different labels

of segments joining vertices in $\mathscr{A}$ (resp. $\mathscr{B}$) does not exceed $|\mathscr{A}|-1$ (resp. $|\mathscr{B}|-1$), while all segments joining a vertex in $\mathscr{A}$ and a vertex in $\mathscr{B}$ are labelled by r. Therefore $r \leq (|\mathscr{A}|-1)+(|\mathscr{B}|-1)+1=n-1$. The equality is achieved if all the mentioned labels are different.

(b) Let a_n be the number of labellings with $r=n-1$. We prove by induction that $a_n=\frac{n!(n-1)!}{2^{n-1}}$. This is trivial for $n=1$, so let $n \geq 2$. If $|\mathscr{A}|=k$ is fixed, the groups $\mathscr{A}$ and $\mathscr{B}$ can be chosen in $\binom{n}{k}$ ways. The set of labels used within $\mathscr{A}$ can be selected among $1,2,\ldots,n-2$ in $\binom{n-2}{k-1}$ ways. Now the segments within groups $\mathscr{A}$ and $\mathscr{B}$ can be labelled so as to satisfy (ii) in a_k and a_{n-k} ways, respectively. This way every labelling has been counted twice, since choosing $\mathscr{A}$ is equivalent to choosing $\mathscr{B}$. It follows that

$$\begin{aligned} a_n &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\binom{n-2}{k-1}a_k a_{n-k} \\ &= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{a_k}{k!(k-1)!}\cdot\frac{a_{n-k}}{(n-k)!(n-k-1)!} \\ &= \frac{n!(n-1)!}{2(n-1)}\sum_{k=1}^{n-1}\frac{1}{2^{k-1}}\cdot\frac{1}{2^{n-k-1}} = \frac{n!(n-1)!}{2^{n-1}}. \end{aligned}$$

10. Denote by L the leftmost and by R the rightmost marker. To start with, note that the parity of the number of black-side-up markers remains unchanged. Hence, if only two markers remain, these markers must have the same color up.

We 'll show by induction on n that the game can be successfully finished if and only if $n \equiv 0$ or $n \equiv 2 \pmod 3$, and that the upper sides of L and R will be black in the first case and white in the second case.

The statement is clear for $n=2,3$. Assume that we finished the game for some n, and denote by k the position of the marker X (counting from the left) that was last removed. Having finished the game, we have also finished the subgames with the k markers from L to X and with the $n-k+1$ markers from X to R (inclusive). Thereby, before X was removed, the upper side of L had been black if $k \equiv 0$ and white if $k \equiv 2 \pmod 3$, while the upper side of R had been black if $n-k+1 \equiv 0$ and white if $n-k+1 \equiv 2 \pmod 3$. Markers L and R were reversed upon the removal of X. Therefore, in the final position L and R are white if and only if $k \equiv n-k+1 \equiv 0$, which yields $n \equiv 2 \pmod 3$, and black if and only if $k \equiv n-k+1 \equiv 2$, which yields $n \equiv 0 \pmod 3$.

On the other hand, a game with n markers can be reduced to a game with $n-3$ markers by removing the second, fourth, and third marker in this order. This finishes the induction.

Second solution. An invariant can be defined as follows. To each white marker with k black markers to its left we assign the number $(-1)^k$. Let S be the sum of the assigned numbers. Then it is easy to verify that the remainder of S modulo 3 remains unchanged throughout the game: For example, when a white marker with two white neighbors and k black markers to its left is removed, S decreases by $3(-1)^t$.

Initially, $S = n$. In the final position with two markers remained S equals 0 if the two markers are black and 2 if these are white (note that, as before, the two markers must be of the same color). Thus $n \equiv 0$ or 2 (mod 3).
Conversely, a game with n markers is reduced to $n-3$ markers as in the first solution.

11. Assume there were n contestants, a_i of whom solved exactly i problems, where $a_0 + \cdots + a_5 = n$. Let us count the number N of pairs (C,P), where contestant C solved the pair of problems P. Each of the 15 pairs of problems was solved by at least $\frac{2n+1}{5}$ contestants, implying $N \geq 15 \cdot \frac{2n+1}{5} = 6n+3$. On the other hand, a_i students solved $\frac{i(i-1)}{2}$ pairs; hence
$$6n+3 \leq N \leq a_2 + 3a_3 + 6a_4 + 10a_5 = 6n + 4a_5 - (3a_3 + 5a_2 + 6a_1 + 6a_0).$$
Consequently $a_5 \geq 1$. Assume that $a_5 = 1$. Then we must have $N = 6n+4$, which is only possible if 14 of the pairs of problems were solved by exactly $\frac{2n+1}{5}$ students and the remaining one by $\frac{2n+1}{5} + 1$ students, and all students but the winner solved 4 problems.
The problem t not solved by the winner will be called *tough* and the pair of problems solved by $\frac{2n+1}{5} + 1$ students *special*.
Let us count the number M_p of pairs (C,P) for which P contains a fixed problem p. Let b_p be the number of contestants who solved p. Then $M_t = 3b_t$ (each of the b_t students solved three pairs of problems containing t), and $M_p = 3b_p + 1$ for $p \neq t$ (the winner solved four such pairs). On the other hand, each of the five pairs containing p was solved by $\frac{2n+1}{5}$ or $\frac{2n+1}{5} + 1$ students, so $M_p = 2n+2$ if the special pair contains p, and $M_p = 2n+1$ otherwise.
Now since $M_t = 3b_t = 2n+1$ or $2n+2$, we have $2n+1 \equiv 0$ or 2 (mod 3). But if $p \neq t$ is a problem not contained in the special pair, we have $M_p = 3b_p + 1 = 2n+1$; hence $2n+1 \equiv 1$ (mod 3), which is a contradiction.

12. Suppose that there exist desired permutations σ and τ for some sequence $a_1, \dots, a_n$. Given a sequence (b_i) with sum divisible by n which differs modulo n from (a_i) only in two positions, say i_1 and i_2, we show how to construct desired permutations σ' and τ' for sequence (b_i). In this way, starting from an arbitrary sequence (a_i) for which σ and τ exist, we can construct desired permutations for any other sequence with sum divisible by n. All congruences below are modulo n.
We know that $\sigma(i) + \tau(i) \equiv b_i$ for all $i \neq i_1, i_2$. We construct the sequence $i_1, i_2, i_3, \dots$ as follows: for each $k \geq 2$, i_{k+1} is the unique index such that
$$\sigma(i_{k-1}) + \tau(i_{k+1}) \equiv b_{i_k}. \qquad (*)$$
Let $i_p = i_q$ be the repetition in the sequence with the smallest q. We claim that $p = 1$ or $p = 2$. Assume on the contrary that $p > 2$. Summing up $(*)$ for $k = p, p+1, \dots, q-1$ and taking the equalities $\sigma(i_k) + \tau(i_k) = b_{i_k}$ for $i_k \neq i_1, i_2$ into account we obtain $\sigma(i_{p-1}) + \sigma(i_p) + \tau(i_{q-1}) + \tau(i_q) \equiv b_p + b_{q-1}$. Since $i_q = i_p$, it

follows that $\sigma(i_{p-1})+\tau(i_{q-1}) \equiv b_{q-1}$ and therefore $i_{p-1}=i_{q-1}$, a contradiction. Thus $p=1$ or $p=2$ as claimed.

Now we define the following permutations:

$$\begin{aligned}
&\sigma'(i_k)=\sigma(i_{k-1}) \text{ for } k=2,3,\dots,q-1 \text{ and } \sigma'(i_1)=\sigma(i_{q-1}),\\
&\tau'(i_k)=\tau(i_{k+1}) \text{ for } k=2,3,\dots,q-1 \text{ and } \tau'(i_1)=\begin{cases}\tau(i_2) \text{ if } p=1,\\ \tau(i_1) \text{ if } p=2;\end{cases}\\
&\sigma'(i)=\sigma(i) \text{ and } \tau'(i)=\tau(i) \quad \text{for } i \notin \{i_1,\dots,i_{q-1}\}.
\end{aligned}$$

Permutations σ' and τ' have the desired property. Indeed, $\sigma'(i)+\tau'(i)=b_i$ obviously holds for all $i \neq i_1$, but then it must also hold for $i=i_1$.

13. For every green diagonal d, let C_d denote the number of green-red intersection points on d. The task is to find the maximum possible value of the sum $\sum_d C_d$ over all green diagonals.

Let d_i and d_j be two green diagonals and let the part of polygon M lying between d_i and d_j have m vertices. There are at most $n-m-1$ red diagonals intersecting both d_i and d_j, while each of the remaining $m-2$ diagonals meets at most one of d_i, d_j. It follows that

$$C_{d_i}+C_{d_j} \leq 2(n-m-1)+(m-2)=2n-m-4. \qquad (*)$$

We now arrange the green diagonals in a sequence $d_1, d_2, \dots, d_{n-3}$ as follows. It is easily seen that there are two green diagonals d_1 and d_2 that divide M into two triangles and an $(n-2)$-gon; then there are two green diagonals d_3 and d_4 that divide the $(n-2)$-gon into two triangles and an $(n-4)$-gon, and so on. We continue this procedure until we end up with a triangle or a quadrilateral. Now the part of M between d_{2k-1} and d_{2k} has at least $n-2k$ vertices for $1 \leq k \leq r$, where $n-3=2r+e$, $e \in \{0,1\}$; hence, by $(*)$, $C_{d_{2k-1}}+C_{d_{2k}} \leq n+2k-4$. Moreover, $C_{d_{n-3}} \leq n-3$. Summing up yields

$$\begin{aligned}
C_{d_1}+C_{d_2}+\cdots+C_{d_{n-3}} &\leq \sum_{k=1}^{r}(n+2k-4)+e(n-3)\\
&=3r^2+e(3r+1)=\left\lceil \frac{3}{4}(n-3)^2 \right\rceil.
\end{aligned}$$

This value is attained in the following example. Let $A_1A_2\dots A_n$ be the n-gon M and let $l=\left[\frac{n}{2}\right]+1$. The diagonals A_1A_i, $i=3,\dots,l$ and A_lA_j, $j=l+2,\dots,n$ are colored in green, whereas the diagonals A_2A_i, $i=l+1,\dots,n$, and $A_{l+1}A_j$, $j=3,\dots,l-1$ are colored in red.

Thus the answer is $\lceil \frac{3}{4}(n-3)^2 \rceil$.

14. Let F be the point of tangency of the incircle with AC and let M and N be the respective points of tangency of AB and BC with the corresponding excircles. If I is the incenter and I_a and P respectively the center and the tangency point with ray AC of the excircle corresponding to A, we have $\frac{AI}{IL}=\frac{AI}{IF}=\frac{AI_a}{I_aP}=\frac{AI_a}{I_aN}$, which implies that $\triangle AIL \sim \triangle AI_aN$. Thus L lies on AN, and analogously K lies on CM. Denote $x=AF$ and $y=CF$. Since $BD=BE$, $AD=BM=x$, and $CE=BN=y$,

the condition $AB+BC=3AC$ gives us $DM=y$ and $EN=x$. Now the triangles CLN and MKA are congruent since their altitudes KD and LE satisfy $DK=EL$, $DM=CE$, and $AD=EN$. Thus $\angle AKM=\angle CLN$, implying that $ACKL$ is cyclic.

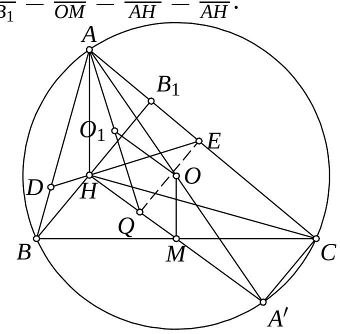
15. Let P be the fourth vertex of the rhombus $C_2A_1A_2P$. Since $\triangle C_2PC_1$ is equilateral, we easily conclude that $B_1B_2C_1P$ is also a rhombus. Thus $\triangle PB_1A_2$ is equilateral and $\angle(C_2A_1,C_1B_2)=\angle A_2PB_1=60^\circ$. It easily follows that $\triangle AC_1B_2\cong\triangle BA_1C_2$ and consequently $AC_1=BA_1$; similarly $BA_1=CB_1$. Therefore triangle $A_1B_1C_1$ is equilateral. Now it follows from $B_1B_2=B_2C_1$ that A_1B_2 bisects $\angle C_1A_1B_1$. Similarly, B_1C_2 and C_1A_2 bisect $\angle A_1B_1C_1$ and $\angle B_1C_1A_1$; hence A_1B_2, B_1C_2, C_1A_2 meet at the incenter of $A_1B_1C_1$, i.e. at the center of ABC.

16. Since $\angle ADL=\angle KBA=180^\circ-\frac{1}{2}\angle BCD$ and $\angle ALD=\frac{1}{2}\angle AYD=\angle KAB$, triangles ABK and LDA are similar. Thus $\frac{BK}{BC}=\frac{BK}{AD}=\frac{AB}{DL}=\frac{DC}{DL}$, which together with $\angle LDC=\angle CBK$ gives us $\triangle LDC\sim\triangle CBK$. Therefore $\angle KCL=360^\circ-\angle BCD-(\angle LCD+\angle KCB)=360^\circ-\angle BCD-(\angle CKB+\angle KCB)=180^\circ-\angle CBK$, which is constant.

17. To start with, we note that points B,E,C are the images of D,F,A respectively under the rotation around point O for the angle $\omega=\angle DOB$, where O is the intersection of the perpendicular bisectors of AC and BD. Then $OE=OF$ and $\angle OFE=\angle OAC=90-\frac{\omega}{2}$; hence the points A,F,R,O are on a circle and $\angle ORP=180^\circ-\angle OFA$. Analogously, the points B,E,Q,O are on a circle and $\angle OQP=180^\circ-\angle OEB=\angle OEC=\angle OFA$. This shows that $\angle ORP=180^\circ-\angle OQP$, i.e. the point O lies on the circumcircle of $\triangle PQR$, thus being the desired point.

18. Let O and O_1 be the circumcenters of triangles ABC and ADE, respectively. It is enough to show that $HM\parallel OO_1$. Let AA' be the diameter of the circumcircle of ABC. We note that if B_1 is the foot of the altitude from B, then HE bisects $\angle CHB_1$. Since the triangles COM and CHB_1 are similar (indeed, $\angle CHB=\angle COM=\angle A$), we have $\frac{CE}{EB_1}=\frac{CH}{HB_1}=\frac{CO}{OM}=\frac{2CO}{AH}=\frac{A'A}{AH}$.

Thus, if Q is the intersection point of the bisector of $\angle A'AH$ with HA', we obtain $\frac{CE}{EB_1}=\frac{A'Q}{QH}$, which together with $A'C\perp AC$ and $HB_1\perp AC$ gives us $QE\perp AC$. Analogously, $QD\perp AB$. Therefore AQ is a diameter of the circumcircle of $\triangle ADE$ and O_1 is the midpoint of AQ. It follows that OO_1 is a middle line in $\triangle A'AQ$ which is parallel to HM.



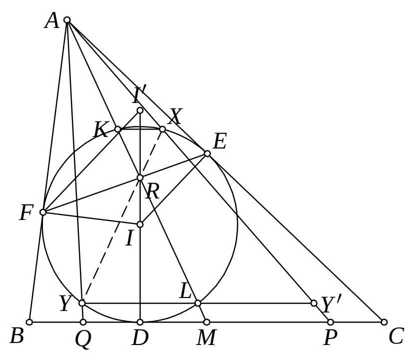
Second solution. We again prove that $OO_1\parallel HM$. Since $AA'=2AO$, it suffices to prove $AQ=2AO_1$.
Elementary calculations of angles give us $\angle ADE=\angle AED=90^\circ-\frac{\alpha}{2}$. Applying the law of sines to $\triangle DAH$ and $\triangle EAH$ we now have $DE=DH+EH=\frac{AH\cos\beta}{\cos\frac{\alpha}{2}}+$

$\frac{AH\cos\gamma}{\cos\frac{\alpha}{2}}$. Since $AH = 2OM = 2R\cos\alpha$, we obtain

$$AO_1 = \frac{DE}{2\sin\alpha} = \frac{AH(\cos\beta + \cos\gamma)}{2\sin\alpha\cos\frac{\alpha}{2}} = \frac{2R\cos\alpha\sin\frac{\alpha}{2}\cos(\frac{\beta-\gamma}{2})}{\sin\alpha\cos\frac{\alpha}{2}}.$$

We now calculate AQ. Let N be the intersection of AQ with the circumcircle. Since $\angle NAO = \frac{\beta-\gamma}{2}$, we have $AN = 2R\cos(\frac{\beta-\gamma}{2})$. Noting that $\triangle QAH \sim \triangle QNM$ (and that $MN = R - OM$), we have

$$AQ = \frac{AN\cdot AH}{MN + AH} = \frac{2R\cos(\frac{\beta-\gamma}{2})\cdot 2\cos\alpha}{1+\cos\alpha} = \frac{2R\cos(\frac{\beta-\gamma}{2})\cos\alpha}{\cos^2\frac{\alpha}{2}} = 2AO_1.$$



19. We denote by D, E, F the points of tangency of the incircle with BC, CA, AB, respectively, by I the incenter, and by Y' the intersection of AX and LY. Since EF is the polar line to the point A with respect to the incircle, it meets AL at point R such that $A, R; K, L$ are conjugated, i.e. $\frac{KR}{RL} = \frac{KA}{AL}$. Then $\frac{KX}{LY'} = \frac{KA}{AL} = \frac{KR}{RL} = \frac{KX}{L\overline{Y}}$ and therefore $LY = L\overline{Y}$, where $\overline{Y}$ is the intersection of XR and LY. Thus showing that $LY = LY'$ (which is the same as showing that $PM = MQ$, i.e. $CP = QB$) is equivalent to showing that XY contains R. Since $XKYL$ is an inscribed trapezoid, it is enough to show that R lies on its axis of symmetry, that is, DI.
Since AM is the median, the triangles ARB and ARC have equal areas and since $\angle(RF, AB) = \angle(RE, AC)$ we have that $1 = \frac{S_{\triangle ABR}}{S_{\triangle ACR}} = \frac{(AB\cdot FR)}{(AC\cdot ER)}$. Hence $\frac{AB}{AC} = \frac{ER}{FR}$.
Let I' be the point of intersection of the line through F parallel to IE with the line IR. Then $\frac{FI'}{EI} = \frac{FR}{RE} = \frac{AC}{AB}$ and $\angle I'FI = \angle BAC$ (angles with orthogonal rays). Thus the triangles ABC and FII' are similar, implying that $\angle FII' = \angle ABC$. Since $\angle FID = 180^\circ - \angle ABC$, it follows that R, I, and D are collinear.

20. We shall show the inequalities $p(ABC) \geq 2p(DEF)$ and $p(PQR) \geq \frac{1}{2}p(DEF)$. The statement of the problem will immediately follow.
Let D_b and D_c be the reflections of D in AB and AC, and let A_1, B_1, C_1 be the midpoints of BC, CA, AB, respectively. It is easy to see that D_b, F, E, D_c are collinear. Hence $p(DEF) = D_bF + FE + ED_c = D_bD_c \leq D_bC_1 + C_1B_1 + B_1D_c = \frac{1}{2}(AB + BC + CA) = \frac{1}{2}p(ABC)$.
To prove the second inequality we observe that P, Q, and R are the points of tangency of the excircles with the sides of $\triangle DEF$. Let $FQ = ER = x$, $DR = FP = y$, and $DQ = EP = z$, and let $\delta, \varepsilon, \varphi$ be the angles of $\triangle DEF$ at D, E, F, respectively. Let Q' and R' be the projections of Q and R onto EF, respectively. Then $QR \geq Q'R' = EF - FQ' - R'E = EF - x(\cos\varphi + \cos\varepsilon)$. Summing this with the analogous inequalities for FD and DE we obtain

$$p(PQR) \geq p(DEF) - x(\cos\varphi + \cos\varepsilon) - y(\cos\delta + \cos\varphi) - z(\cos\delta + \cos\varepsilon).$$

Assuming w.l.o.g. that $x \leq y \leq z$ we also have $DE \leq FD \leq FE$ and consequently $\cos\varphi + \cos\varepsilon \geq \cos\delta + \cos\varphi \geq \cos\delta + \cos\varepsilon$. Now Chebyshev's inequality gives us $p(PQR) \geq p(DEF) - \frac{2}{3}(x+y+z)(\cos\varepsilon + \cos\varphi + \cos\delta) \geq p(DEF) - (x+y+z) = \frac{1}{2}p(DEF)$, where we used $x+y+z = \frac{1}{2}p(DEF)$ and the fact that the sum of the cosines of the angles in a triangle does not exceed $\frac{3}{2}$. This finishes the proof.

21. We will show that 1 is the only such number. It is sufficient to prove that for every prime number p there exists some a_m such that $p \mid a_m$. For $p = 2,3$ we have $p \mid a_2 = 48$. Assume now that $p > 3$. Appyling Fermat's theorem, we have:

$$6a_{p-2} = 3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} - 6 \equiv 3 + 2 + 1 - 6 = 0 \pmod{p}.$$

Hence $p \mid a_{p-2}$, i.e. $\gcd(p, a_{p-2}) = p > 1$. This completes the proof.

22. It immediately follows from the condition of the problem that all the terms of the sequence are distinct. We also note that $|a_i - a_n| \leq n-1$ for all integers i,n where $i < n$, because if $d = |a_i - a_n| \geq n$ then $\{a_1, \dots, a_d\}$ contains two elements congruent to each other modulo d, which is a contradiction. It easily follows by induction that for every $n \in \mathbb{N}$ the set $\{a_1, \dots, a_n\}$ consists of consecutive integers. Thus, if we assumed some integer k did not appear in the sequence $a_1, a_2, \dots$, the same would have to hold for all integers either larger or smaller than k, which contradicts the condition that infinitely many positive and negative integers appear in the sequence. Thus, the sequence contains all integers.

23. Let us consider the polynomial

$$P(x) = (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f) = Sx^2 + Qx + R,$$

where $Q = ab + bc + ca - de - ef - fd$ and $R = abc + def$.
Since $S \mid Q, R$, it follows that $S \mid P(x)$ for every $x \in \mathbb{Z}$. Hence, $S \mid P(d) = (d+a)(d+b)(d+c)$. Since $S > d+a, d+b, d+c$ and thus cannot divide any of them, it follows that S must be composite.

24. We will show that n has the desired property if and only if it is prime.
For $n = 2$ we can take only $a = 1$. For $n > 2$ and even, $4 \mid n!$, but $a^n + 1 \equiv 1, 2 \pmod{4}$, which is impossible. Now we assume that n is odd. Obviously $(n!-1)^n + 1 \equiv (-1)^n + 1 = 0 \pmod{n!}$. If n is composite and d its prime divisor, then $\left(\frac{n!}{d} - 1\right)^n + 1 = \sum_{k=1}^{n} \binom{n}{k} \frac{n!^k}{d^k}$, where each summand is divisible by $n!$ because $d^2 \mid n!$; therefore $n!$ divides $\left(\frac{n!}{d} - 1\right)^n + 1$. Thus, all composite numbers are ruled out.
It remains to show that if n is an odd prime and $n! \mid a^n + 1$, then $n! \mid a + 1$ and therefore $a = n! - 1$ is the only relevant value for which $n! \mid a^n + 1$. Consider any prime number $p \leq n$. If $p \mid \frac{a^n+1}{a+1}$, we have $p \mid (-a)^n - 1$ and by Fermat's theorem $p \mid (-a)^{p-1} - 1$. Therefore $p \mid (-a)^{(n,p-1)} - 1 = -a - 1$, i.e. $a \equiv -1 \pmod{p}$. But then $\frac{a^n+1}{a+1} = a^{n-1} - a^{n-2} + \cdots - a + 1 \equiv n \pmod{p}$, implying that $p = n$. It

follows that $\frac{a^n+1}{a+1}$ is coprime to $(n-1)!$ and consequently $(n-1)!$ divides $a+1$. Moreover, the above consideration shows that n must divide $a+1$. Thus $n! \mid a+1$ as claimed. This finishes our proof.

25. We will use the abbreviation HD to denote a "highly divisible integer". Let $n = 2^{\alpha_2(n)}3^{\alpha_3(n)}\cdots p^{\alpha_p(n)}$ be the factorization of n into primes. We have $d(n) = (\alpha_2(n)+1)\cdots(\alpha_p(n)+1)$. We start with the following two lemmas.

Lemma 1. If n is a HD and p,q primes with $p^k < q^l$ ($k,l \in \mathbb{N}$), then

$$k\alpha_q(n) \le l\alpha_p(n) + (k+1)(l-1).$$

Proof. The inequality is trivial if $\alpha_q(n) < l$. Suppose that $\alpha_q(n) \ge l$. Then np^k/q^l is an integer less than q, and $d(np^k/q^l) < d(n)$, which is equivalent to $(\alpha_q(n)+1)(\alpha_p(n)+1) > (\alpha_q(n)-l+1)(\alpha_p(n)+k+1)$ implying the desired inequality.

Lemma 2. For each p and k there exist only finitely many HD's n such that $\alpha_p(n) \le k$.

Proof. It follows from Lemma 1 that if n is a HD with $\alpha_p(n) \le k$, then $\alpha_q(n)$ is bounded for each prime q and $\alpha_q(n) = 0$ for $q > p^{k+1}$. Therefore there are only finitely many possibilities for n.

We are now ready to prove both parts of the problem.

(a) Suppose that there are infinitely many pairs (a,b) of consecutive HD's with $a \mid b$. Since $d(2a) > d(a)$, we must have $b = 2a$. In particular, $d(s) \le d(a)$ for all $s < 2a$. All but finitely many HD's a are divisible by 2 and by 3^7. Then $d(8a/9) < d(a)$ and $d(3a/2) < d(a)$ yield

$$(\alpha_2(a)+4)(\alpha_3(a)-1) < (\alpha_2(a)+1)(\alpha_3(a)+1) \Rightarrow 3\alpha_3(a)-5 < 2\alpha_2(a),$$
$$\alpha_2(a)(\alpha_3(a)+2) \le (\alpha_2(a)+1)(\alpha_3(a)+1) \Rightarrow \alpha_2(a) \le \alpha_3(a)+1.$$

We now have $3\alpha_3(a)-5 < 2\alpha_2(a) \le 2\alpha_3(a)+2 \Rightarrow \alpha_3(a) < 7$, which is a contradiction.

(b) Assume for a given prime p and positive integer k that n is the smallest HD with $\alpha_p \ge k$. We show that $\frac{n}{p}$ is also a HD. Assume the opposite, i.e. that there exists a HD $m < \frac{n}{p}$ such that $d(m) \ge d(\frac{n}{p})$. By assumption, m must also satisfy $\alpha_p(m)+1 \le \alpha_p(n)$. Then

$$d(mp) = d(m)\frac{\alpha_p(m)+2}{\alpha_p(m)+1} \ge d(n/p)\frac{\alpha_p(n)+1}{\alpha_p(n)} = d(n),$$

contradicting the initial assumption that n is a HD (since $mp < n$). This proves that $\frac{n}{p}$ is a HD. Since this is true for every positive integer k the proof is complete.

26. Assuming $b \ne a$, it trivially follows that $b > a$. Let $p > b$ be a prime number and let $n = (a+1)(p-1)+1$. We note that $n \equiv 1 \pmod{p-1}$ and $n \equiv -a \pmod{p}$. It follows that $r^n = r\cdot(r^{p-1})^{a+1} \equiv r \pmod{p}$ for every integer r. We now have $a^n + n \equiv a - a = 0 \pmod{p}$. Thus, a^n+n is divisible by p, and hence by the condition

of the problem $b^n + n$ is also divisible by p. However, we also have $b^n + n \equiv b - a \pmod{p}$, i.e. $p \,|\, b-a$, which contradicts $p > b$. Hence, it must follow that $b = a$. We note that $b = a$ trivially fulfills the conditions of the problem for all $a \in \mathbb{N}$.

27. Let p be a prime and $k < p$ an even number. We note that $(p-k)!(k-1)! \equiv (-1)^{k-1}(p-k)!(p-k+1)\dots(p-1) = (-1)^{k-1}(p-1)! \equiv 1 \pmod{p}$ by Wilson's theorem. Therefore

$$\begin{aligned}(k-1)!^n P((p-k)!) &= \textstyle\sum_{i=0}^n a_i[(k-1)!]^{n-i}[(p-k)!(k-1)!]^i\\ &\equiv \textstyle\sum_{i=0}^n a_i[(k-1)!]^{n-i} = S((k-1)!) \pmod{p},\end{aligned}$$

where $S(x) = a_n + a_{n-1}x + \cdots + a_0x^n$. Hence $p \mid P((p-k)!)$ if and only if $p \mid S((k-1)!)$. Note that $S((k-1)!)$ depends only on k. Let $k > 2a_n + 1$. Then, $s = (k-1)!/a_n$ is an integer which is divisible by all primes smaller than k. Hence $S((k-1)!) = a_n b_k$ for some $b_k \equiv 1 \pmod{s}$. It follows that b_k is divisible only by primes larger than k. For large enough k we have $|b_k| > 1$. Thus for every prime divisor p of b_k we have $p \mid P((p-k)!)$.
It remains to select a large enough k for which $|P((p-k)!)| > p$. We take $k = (q-1)!$, where q is a large prime. All the numbers $k+i$ for $i = 1, 2, \dots, q-1$ are composite (by Wilson's theorem, $q \mid k+1$). Thus $p = k+q+r$, for some $r \geq 0$. We now have $|P((p-k)!)| = |P((q+r)!)| > (q+r)! > (q-1)! + q + r = p$, for large enough q, since $n = \deg P \geq 2$. This completes the proof.
Remark. The above solution actually also works for all linear polynomials P other than $P(x) = x + a_0$. Nevertheless, these particular cases are easily handled. If $|a_0| > 1$, then $P(m!)$ is composite for $m > |a_0|$, whereas $P(x) = x + 1$ and $P(x) = x - 1$ are both composite for, say, $x = 5!$. Thus the condition $n \geq 2$ was redundant.

A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathscr{B}(A,B,C)$, $A-B-C$: indicates the relation of *betweenness*, i.e., that B is between A and C (this automatically means that A,B,C are different collinear points).
- $A = l_1 \cap l_2$: indicates that A is the intersection point of the lines l_1 and l_2.
- AB: line through A and B, segment AB, length of segment AB (depending on context).
- $[AB$: ray starting in A and containing B.
- $(AB$: ray starting in A and containing B, but without the point A.
- (AB): open interval AB, set of points between A and B.
- $[AB]$: closed interval AB, segment AB, $(AB) \cup \{A,B\}$.
- $(AB]$: semiopen interval AB, closed at B and open at A, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- ABC: plane determined by points A,B,C, triangle ABC ($\triangle ABC$) (depending on context).
- $[AB,C$: half-plane consisting of line AB and all points in the plane on the same side of AB as C.
- $(AB,C$: $[AB,C$ without the line AB.

- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle ABC (unless otherwise indicated).
- $k(O, r)$: circle k with center O and radius r.
- $d(A, p)$: distance from point A to line p.
- $S_{A_1A_2\ldots A_n}$: area of n-gon $A_1A_2\ldots A_n$ (special case for $n = 3$, S_{ABC}: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo n, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo p, p being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in x with integer and real coefficients respectively.
- R^*: the set of nonzero elements of a ring R.
- $R[\alpha]$, $R(\alpha)$, where α is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- X_0: $X \cup \{0\}$ for X such that $0 \notin X$.
- X^+, X^-, $aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to x.
- $\lceil x \rceil$: the smallest integer greater than or equal to x.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points AB, the absolute value of the number x, the number of elements of the set S (respectively).
- (x, y), (m, n), (a, b): (ordered) pair x and y, the greatest common divisor of integers m and n, the open interval between real numbers a and b (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).
- LHS: left-hand side (of a given equation).

- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

B

Codes of the Countries of Origin

ARG	Argentina	HKG	Hong Kong	POL	Poland
ARM	Armenia	HUN	Hungary	POR	Portugal
AUS	Australia	ICE	Iceland	PRK	Korea, North
AUT	Austria	INA	Indonesia	PUR	Puerto Rico
BEL	Belgium	IND	India	ROM	Romania
BLR	Belarus	IRE	Ireland	RUS	Russia
BRA	Brazil	IRN	Iran	SAF	South Africa
BUL	Bulgaria	ISR	Israel	SER	Serbia
CAN	Canada	ITA	Italy	SIN	Singapore
CHN	China	JAP	Japan	SLO	Slovenia
COL	Colombia	KAZ	Kazakhstan	SMN	Serbia and Montenegro
CRO	Croatia	KOR	Korea, South	SPA	Spain
CUB	Cuba	KUW	Kuwait	SVK	Slovakia
CYP	Cyprus	LAT	Latvia	SWE	Sweden
CZE	Czech Republic	LIT	Lithuania	THA	Thailand
CZS	Czechoslovakia	LUX	Luxembourg	TUN	Tunisia
EST	Estonia	MCD	Macedonia	TUR	Turkey
FIN	Finland	MEX	Mexico	TWN	Taiwan
FRA	France	MON	Mongolia	UKR	Ukraine
FRG	Germany, FR	MOR	Morocco	USA	United States
GBR	United Kingdom	NET	Netherlands	USS	Soviet Union
GDR	Germany, DR	NOR	Norway	UZB	Uzbekistan
GEO	Georgia	NZL	New Zealand	VIE	Vietnam
GER	Germany	PER	Peru	YUG	Yugoslavia
GRE	Greece	PHI	Philippines		