Dušan Djukić Vladimir Janković
Ivan Matić Nikola Petrović

IMO Shortlist 2006

From the book "The IMO Compendium"

www.imomath.com

Springer

1

Problems

1.1 The Forty-Seventh IMO
Ljubljana, Slovenia, July 6–18, 2006

1.1.1 Contest Problems

First Day (July 12)

1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies
$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$
Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

2. Let $\mathscr{P}$ be a regular 2006-gon. A diagonal of $\mathscr{P}$ is called *good* if its endpoints divide the boundary of $\mathscr{P}$ into two parts, each composed of an odd number of sides of $\mathscr{P}$. The sides of $\mathscr{P}$ are also called good.
Suppose $\mathscr{P}$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $\mathscr{P}$. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

3. Determine the least real number M such that the inequality
$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \leq M(a^2+b^2+c^2)^2$$
holds for all real numbers a, b and c.

Second Day (July 13)

4. Determine all pairs (x,y) of integers such that
$$1+2^x+2^{2x+1}=y^2.$$

5. Let $P(x)$ be a polynomial of degree $n>1$ with integer coefficients and let k be a positive integer. Consider the polynomial

$$Q(x) = P(P(\dots P(P(x))\dots)),$$

where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

6. Assign to each side b of a convex polygon $\mathscr{P}$ the maximum area of a triangle that has b as a side and is contained in $\mathscr{P}$. Show that the sum of the areas assigned to the sides of $\mathscr{P}$ is at least twice the area of $\mathscr{P}$.

1.1.2 Shortlisted Problems

1. **A1 (EST)** A sequence of real numbers $a_0, a_1, a_2, \dots$ is defined by the formula
$$a_{i+1} = [a_i] \cdot \{a_i\}, \text{ for } i \geq 0;$$
here a_0 is an arbitrary number, $[a_i]$ denotes the greatest integer not exceeding a_i, and $\{a_i\} = a_i - [a_i]$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

2. **A2 (POL)** The sequence of real numbers $a_0, a_1, a_2, \dots$ is defined recursively by
$$a_0 = -1, \ \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \text{ for } n \geq 1.$$
Show that $a_n > 0$ for $n \geq 1$.

3. **A3 (RUS)** The sequence $c_0, c_1, \dots, c_n, \dots$ is defined by $c_0 = 1$, $c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β, and M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality $m < \alpha x + \beta y < M$ if and only if $(x, y) \in S$.
Remark: A sum over the elements of the empty set is assumed to be 0.

4. **A4 (SER)** Prove the inequality
$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$
for positive real numbers $a_1, a_2, \dots, a_n$.

5. **A5 (KOR)** Let a, b, c be the sides of a triangle. Prove that
$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

6. **A6 (IRE)**[IMO3] Determine the smallest number M such that the inequality
$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq M(a^2 + b^2 + c^2)^2$$
holds for all real numbers a, b, c

7. **C1 (FRA)** We have $n \geq 2$ lamps $L_1, \dots, L_n$ in a row, each of them being either *on* or *off*. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; – otherwise, L_i is switched on.
   Initially all the lamps are off except the leftmost one which is on.
   (a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.
   (b) Prove that there are infinitely many integers n for which the lamps will never be all off.

8. **C2 (SER)**[IMO2] A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals. Suppose the 2006-gon has been dissected into triangles by 2003 non-intersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

9. **C3 (COL)** Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

   where the sum is taken over all convex polygons with vertices in S.
   Remark. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively.

10. **C4 (TWN)** A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathscr{A}$.
    Let $\mathscr{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathscr{B}$ than of arrangement $\mathscr{A}$. Prove that arrangement $\mathscr{B}$ can be obtained from $\mathscr{A}$ by performing a number of *switches*, defined as follows:
    A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

11. **C5 (ARG)** An (n,k)-tournament is a contest with n players held in k rounds such that:
    (i) Each player plays in each round, and every two players meet at most once.
    (ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.
    Determine all pairs (n,k) for which there exists an (n,k)-tournament.

12. **C6 (COL)** A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A *diamond* is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

13. **C7 (JAP)** Consider a convex polyheadron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyheadron *antipodal* if there exist two parallel planes passing through these points and such that the polyheadron is contained between these planes.
Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

14. **G1 (KOR)**[IMO1] Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$ and that equality holds if and only if P coincides with I.

15. **G2 (UKR)** Let ABC be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B, and C are concyclic.

16. **G3 (USA)** Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

17. **G4 (RUS)** A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

18. **G5 (GRE)** In triangle ABC, let J be the center of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

19. **G6 (BRA)** Circles ω_1 and ω_2 with centers O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F, repsectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, and O_1 are on the same side of t. Prove that the lines AO_1, BO_2, EF, and t are concurrent.

20. **G7 (SVK)** In an triangle ABC, let M_a, M_b, M_c, be resepctively the midpoints of the sides BC, CA, AB, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not couning the opposite vertices. For $i \in \{a, b, c\}$ let ω_i be the circle with M_iT_i as diameter. Let p_i be the common external tangent to

ω_j, ω_k ($\{i,j,k\} = \{a,b,c\}$) such that ω_i lies on the opposite side of p_i than ω_j, ω_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

21. **G8 (POL)** Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that $\angle PAB + \angle PDC \leq 90^\circ$ and $\angle PBA + \angle PCD \leq 90^\circ$. Prove that $AB + CD \geq BC + AD$.
22. **G9 (RUS)** Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB, respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.
23. **G10 (SER)**$^{\text{IMO6}}$ Assign to each side b of a convex polygon $\mathscr{P}$ the maximum area of a triangle that has b as a side and is contained in $\mathscr{P}$. Show that the sum of the areas assigned to the sides of $\mathscr{P}$ is at least twice the area of $\mathscr{P}$.
24. **N1 (USA)**$^{\text{IMO4}}$ Determine all pairs (x,y) of integers satisfying the equation $1 + 2^x + 2^{2x+1} = y^2$.
25. **N2 (CAN)** For $x \in (0,1)$ let $y \in (0,1)$ be the number whose nth digit after the decimal point is the 2^nth digit after the decimal point of x. Show that if x is rational then so is y.
26. **N3 (SAF)** The sequence $f(1)$, $f(2)$, $f(3)$, $\dots$ is defined by
$$f(n) = \frac{1}{n}\left(\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right]\right),$$
where $[x]$ denotes the integral part of x.
(a) Prove that $f(n+1) > f(n)$ infinitely often.
(b) Prove that $f(n+1) < f(n)$ infinitely often.
27. **N4 (ROM)**$^{\text{IMO5}}$ Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\dots P(P(x))\dots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.
28. **N5 (RUS)** Find all integer solutions of the equation
$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$
29. **N6 (USA)** Let $a > b > 1$ be relatively prime positive integers. Define the *weight* of an integer c, denoted by $w(c)$ to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that $ax + by = c$. An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$. Find all local champions and determine their number.

30. **N7 (EST)** Prove that for every positive integer n there exists an integer m such that $2^m + m$ is divisible by n.

2

Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2006

1. If $a_0 \ge 0$ then $a_i \ge 0$ for each i and $[a_{i+1}] \le a_{i+1} = [a_i]\{a_i\} < [a_i]$ unless $[a_i] = 0$. Eventually 0 appears in the sequence $[a_i]$ and all subsequent a_k's are 0.
Now suppose that $a_0 < 0$; then all $a_i \le 0$. Suppose that the sequence never reaches 0. Then $[a_i] \le -1$ and so $1 + [a_{i+1}] > a_{i+1} = [a_i]\{a_i\} > [a_i]$, so the sequence $[a_i]$ is nondecreasing and hence must be constant from some term on: $[a_i] = c < 0$ for $i \ge n$. The defining formula becomes $a_{i+1} = c\{a_i\} = c(a_i - c)$ which is equivalent to $b_{i+1} = cb_i$, where $b_i = a_i - \frac{c^2}{c-1}$. Since (b_i) is bounded, we must have either $c = -1$, in which case $a_{i+1} = -a_i - 1$ and hence $a_{i+2} = a_i$, or $b_i = 0$ and thus $a_i = \frac{c^2}{c-1}$ for all $i \ge n$.
2. We use induction on n. We have $a_1 = 1/2$; assume that $n \ge 1$ and $a_1, \dots, a_n > 0$. The formula gives us $(n+1)\sum_{k=1}^{m} \frac{a_k}{m-k+1} = 1$. Writing this equation for n and $n+1$ and subtracting yields
$$(n+2)a_{n+1} = \sum_{k=1}^{n} \left(\frac{n+1}{n-k+1} - \frac{n+2}{n-k+2} \right) a_k$$
which is positive as so is the coefficient at each a_k.
Remark. By using techniques from complex analysis such as contour integrals one can obtain the following formula for $n \ge 1$:
$$a_n = \int_1^{\infty} \frac{dx}{x^n(\pi^2 + \ln^2(x-1))} > 0.$$
3. We know that $c_n = \frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the roots of $t^2 - t - 1$. Since $c_{n-1}/c_n \to -\psi$, taking $\alpha = \psi$ and $\beta = 1$ is a natural choice. For every finite set $J \subseteq \mathbb{N}$ we have
$$-1 = \sum_{n=0}^{\infty} \psi^{2n+1} < \psi x + y = \sum_{j \in J} \psi^{j-1} < \sum_{n=0}^{\infty} \psi^{2n} = \phi.$$
Thus $m = -1$ and $M = \phi$ is an appropriate choice. We now prove that this choice has the desired properties by showing that, for any $x, y \in \mathbb{N}$ with $-1 < K = x\psi + y < \phi$, there is a finite set $J \subset \mathbb{N}$ such that $K = \sum_{j \in J} \psi^j$.
Given such K, there are sequences $i_1 \le \cdots \le i_k$ with $\psi^{i_1} + \cdots + \psi^{i_k} = K$ (one such sequence consists of y zeros and x ones). Consider all such sequences of minimum length n. Since $\psi^m + \psi^{m+1} = \psi^{m+2}$, these sequences contain no two consecutive integers. Order such sequences as follows: If $i_k = j_k$ for $1 \le k \le t$ and $i_t < j_t$, then $(i_r) \prec (j_r)$. Consider the smallest sequence $(i_r)_{r=1}^n$ in this ordering. We claim that its terms are distinct. Since $2\psi^2 = 1 + \psi^3$, replacing two equal terms m, m by $m-2, m+1$ for $m \ge 2$ would yield a smaller sequence, so only 0 or 1 can repeat among the i_r. But $i_t = i_{t+1} = 0$ implies $\sum_r \psi^{i_r} > 2 + \sum_{k=0}^{\infty} \psi^{2k+3} = \phi$, while $i_t = i_{t+1} = 1$ similarly implies $\sum_r \psi^{i_r} < -1$, so both cases are impossible, proving our claim. Thus $J = \{i_1, \dots, i_n\}$ is a required set.

4. Since $\frac{ab}{a+b} = \frac{1}{4}\left(a+b-\frac{(a-b)^2}{a+b}\right)$, the left hand side of the desired inequality equals
$$A = \sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \frac{n-1}{4}\sum_k a_k - \frac{1}{4}\sum_{i<j}\frac{(a_i - a_j)^2}{a_i + a_j}.$$
The right hand side of the inequality is equal to
$$B = \frac{n}{2}\frac{\sum a_i a_j}{\sum a_k} = \frac{n-1}{4}\sum_k a_k - \frac{1}{4}\sum_{i<j}\frac{(a_i - a_j)^2}{\sum a_k}.$$
Now $A \le B$ follows from the trivial inequality $\sum \frac{(a_i-a_j)^2}{a_i+a_j} \ge \sum \frac{(a_i-a_j)^2}{\sum a_k}$.

5. Let $x = \sqrt{b}+\sqrt{c}-\sqrt{a}$, $y = \sqrt{c}+\sqrt{a}-\sqrt{b}$, and $z = \sqrt{a}+\sqrt{b}-\sqrt{c}$. All of these numbers are positive because a,b,c are sides of a triangle. Then $b+c-a = x^2 - \frac{1}{2}(x-y)(x-z)$ and
$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} = \sqrt{1-\frac{(x-y)(y-z)}{2x^2}} \le 1 - \frac{(x-y)(x-z)}{4x^2}.$$
Now it is enough to prove that
$$x^{-2}(x-y)(x-z) + y^{-2}(y-z)(y-x) + z^{-2}(z-x)(z-y) \ge 0$$
which directly follows from Schur's inequality.

6. Assume w.l.o.g. that $a \ge b \ge c$. The LHS of the inequality equals $L = (a-b)(b-c)(a-c)(a+b+c)$. From $(a-b)(b-c) \le \frac{1}{4}(a-c)^2$ we get $L \le \frac{1}{4}(a-c)^3|a+b+c|$. The inequality $(a-c)^2 \le 2(a-b)^2 + 2(b-c)$ implies $(a-c)^2 \le \frac{2}{3}[(a-b)^2+(b-c)^2+(a-c)^2]$. Therefore
$$L \le \frac{\sqrt{2}}{2}\left(\frac{(a-b)^2+(b-c)^2+(a-c)^2}{3}\right)^{3/2}(a+b+c).$$
Finally, the mean inequality gives us
$$\begin{aligned} L &\le \frac{\sqrt{2}}{2}\left(\frac{(a-b)^2+(b-c)^2+(a-c)^2+(a+b+c)^2}{4}\right)^2 \\ &= \frac{9\sqrt{2}}{32}(a^2+b^2+c^2)^2. \end{aligned}$$
The equality is attained if and only if $a-b = b-c$ and $(a-b)^2+(b-c)^2+(a-c)^2 = 3(a+b+c)^2$, which leads to $a = \left(1+\frac{3}{\sqrt{2}}\right)b$ and $c = \left(1-\frac{3}{\sqrt{2}}\right)b$. Thus $M = \frac{9\sqrt{2}}{32}$.
Second solution. We have $L = |(a-b)(b-c)(c-a)(a+b+c)|$. Assume w.l.o.g. that $a+b+c = 1$ (the case $a+b+c=0$ is trivial). The monic cubic polynomial with the roots $a-b$, $b-c$ and $c-a$ is of the form

$$P(x) = x^3 + qx + r, \quad q = \frac{1}{2} - \frac{3}{2}(a^2+b^2+c^2), \ \ r = -(a-b)(b-c)(c-a).$$

Then $M^2 = \max r^2 / \left(\frac{1-2q}{3}\right)^4$. Since $P(x)$ has three real roots, its discriminant $(q/3)^3 + (r/2)^2$ must be positive, so $r^2 \geq -\frac{4}{27}q^3$. Thus $M^2 \leq f(q) = -\frac{4}{27}q^3 / \left(\frac{1-2q}{3}\right)^4$. Function f attains its maximum $3^4/2^9$ at $q = -3/2$, so $M \leq \frac{9\sqrt{2}}{32}$. The case of equality is easily computed.

Third solution. Assume that $a^2+b^2+c^2 = 1$ and write $u = (a+b+c)/\sqrt{3}$, $v = (a+\varepsilon b+\varepsilon^2 c)/\sqrt{3}$, $w = (a+\varepsilon^2 b+\varepsilon c)/\sqrt{3}$, where $\varepsilon = e^{2\pi i/3}$. Then analogous formulas hold for a,b,c in terms of u,v,w, from which one directly obtains $|u|^2 + |v|^2 + |w|^2 = a^2+b^2+c^2 = 1$ and

$$a+b+c = \sqrt{3}u, \quad |a-b| = |v-\varepsilon w|, \ \ |a-c| = |v-\varepsilon^2 w|, \ \ |b-c| = |v-w|.$$

Thus $L = \sqrt{3}|u||v^3-w^3| \leq \sqrt{3}|u|(|v|^3+|w|^3) \leq \sqrt{\frac{3}{2}|u|^2(1-|u|^2)^3} \leq \frac{9\sqrt{2}}{32}$. It is easy to trace back a,b,c to the equality case.

7. (a) We show that for $n = 2^k$ all lamps will be switched on in $n-1$ steps and off in n steps. For $k = 1$ the statement is true. Suppose it holds for some k and let $n = 2^{k+1}$; denote $L = \{L_1, \dots, L_{2^k}\}$ and $R = \{L_{2^k+1}, \dots, L_{2^{k+1}}\}$. The first $2^k - 1$ steps are performed without any influence on or from the lamps from R; thus after $2^k - 1$ steps the lamps in L are on and those from R are off. After the 2^k-th step, L_{2^k} and L_{2^k+1} are on and the other lamps are off. Notice that from now L and R will be symmetric (i.e. L_i and $L_{2^{k+1}-i}$ will have the same state) and will never influence each other. Since R starts with only the leftmost lamp on, in 2^k steps all its lamps will be off. The same will happen to L. There are $2^k + 2^k = 2^{k+1}$ steps in total.

   (b) We claim that for $n = 2^k + 1$ the lamps cannot be switched off. After the first step only L_1 and L_2 are on. According to (a), after $2^k - 1$ steps all lamps but L_n will be on, so after the 2^k-th step all lamps will be off except for L_{n-1} and L_n. Since this position is symmetric to the one after the first step, the procedure will never end.

8. We call a triangle *odd* if it has two odd sides. To any odd isosceles triangle $A_iA_jA_k$ we assign a pair of sides of the 2006-gon. We may assume that $k - j = j - i > 0$ is odd. A side of the 2006-gon is said to *belong* to triangle $A_iA_jA_k$ if it lies on the polygonal line $A_iA_{i+1}\dots A_k$. At least one of the odd number of sides $A_iA_{i+1}, \dots, A_{j-1}A_j$ and at least one of the sides $A_jA_{j+1}, \dots, A_{k-1}A_k$ do not belong to any other odd isosceles triangle; assign those two sides to $\triangle A_iA_jA_k$. This ensures that every two assigned pairs are disjoint; therefore there are at most 1003 odd isosceles triangles.

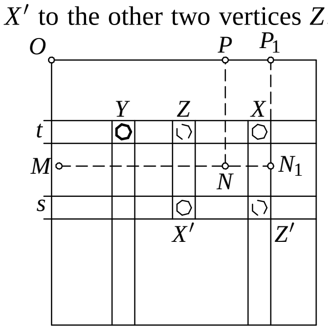
   An example with 1003 odd isosceles triangles can be attained when the diagonals $A_{2k}A_{2k+2}$ are drawn for $k = 0, \dots, 1002$, where $A_0 = A_{2006}$.

9. The number $c(P)$ of points inside P is equal to $n - a(P) - b(P)$, where $n = |S|$. Writing $y = 1 - x$ the considered sum becomes

$$\begin{aligned}\sum_P x^{a(P)}y^{b(P)}(x+y)^{c(P)} &= \sum_P \sum_{i=0}^{c(P)} \binom{c(P)}{i} x^{a(P)+i}y^{b(P)+c(P)-i}\\ &= \sum_P \sum_{k=a(P)}^{a(P)+c(P)} \binom{c(P)}{k-a(P)} x^k y^{n-k}.\end{aligned}$$

Here the coefficient at x^ky^{n-k} is the sum $\sum_P \binom{c(P)}{k-a(P)}$ which equals the number of pairs (P,Z) of a convex polygon P and a k-element subset Z of S whose convex hull is P, and is thus equal to $\binom{n}{k}$. Now the required statement immediately follows.

10. Denote by $S_{\mathscr{A}}(R)$ the number of strawberries of arrangement $\mathscr{A}$ inside rectangle R. We write $\mathscr{A} \le \mathscr{B}$ if for every rectangle Q containing the top left corner O we have $S_{\mathscr{B}}(Q) \ge S_{\mathscr{A}}(Q)$. In this ordering, every switch transforms an arrangement to a larger one. Since the number of arrangements is finite, it is enough to prove that whenever $\mathscr{A} < \mathscr{B}$ there is a switch taking $\mathscr{A}$ to $\mathscr{C}$ with $\mathscr{C} \le \mathscr{B}$. Consider the highest row t of the cake which differs in $\mathscr{A}$ and $\mathscr{B}$; let X and Y be the positions of the strawberries in t in $\mathscr{A}$ and $\mathscr{B}$ respectively. Clearly Y is to the left from X and the strawberry of $\mathscr{A}$ in the column of Y is below Y. Now consider the highest strawberry X' of $\mathscr{A}$ below t whose column is between X and Y (including Y). Let s be the row of X'. Now switch X,X' to the other two vertices Z,Z' of the corresponding rectangle, obtaining an arrangement $\mathscr{C}$. We claim that $\mathscr{C} \le \mathscr{B}$. It is enough to verify that $S_{\mathscr{C}}(Q) \le S_{\mathscr{B}}(Q)$ for those rectangles $Q = OMNP$ with N lying inside $XZX'Z'$. Let $Q' = OMN_1P_1$ be the smallest rectangle containing X. Our choice of s ensures that $S_{\mathscr{C}}(Q) = S_{\mathscr{A}}(Q') \ge S_{\mathscr{B}}(Q') \ge S_{\mathscr{B}}(Q)$, as claimed.



11. Let q be the largest integer such that $2^q \mid n$. We prove that an (n,k)-tournament exists if and only if $k < 2^q$.
The first l rounds of an (n,k)-tournament form an (n,l)-tournament. Thus it is enough to show that a $(n,2^q-1)$-tournament exists and a $(n,2^q)$-tournament does not.
If $n = 2^q$, we can label the contestants and rounds by elements of the additive group $\mathbb{Z}_2^q$. If contestants x and $x+j$ meet in the round labelled j, it is easy to verify the conditions. If $n = 2^qp$, we can divide the contestants into p disjoint groups of 2^q and perform a $(2^q,2^q-1)$-tournament in each, thus obtaining an $(n,2^q-1)$-tournament.
For the other direction let $\mathscr{G}_i$ be the graph of players with edges between any two players who met in the first i rounds. We claim that the size of each connected component of $\mathscr{G}_i$ is a power of 2. For $i = 1$ this is obvious; assume it holds for i. Suppose that the components C and D merge in the $(i+1)$-th round. Then some

$c \in C$ and $d \in D$ meet in this round. Moreover, each player in C meets a player in D. Indeed, for every $c' \in C$ there is a path $c = c_0, c_1, \ldots, c_k = c'$ with $c_j c_{j+1} \in \mathscr{G}_i$; then if d_j is the opponent of c_j in the $(i+1)$-th round, condition (ii) shows that each $d_j d_{j+1} \in \mathscr{G}_i$, so $d_k \in D$. Analogously, all players in D meet players in C, so $|C| = |D|$, proving our claim. Now if there are 2^q rounds, every component has size at least $2^q + 1$ and is thus divisible by 2^{q+1}, which is impossible if $2^{q+1} \nmid n$.

12. Let U and D be the sets of upward and downward unit triangles, respectively. Two triangles are *neighbors* if they form a diamond. For $A \subseteq D$, denote by $F(A)$ the set of neighbors of he elements of A.
If a holey triangle can be tiled with diamonds, in every upward triangle of side l there are l^2 elements of D, so there must be at least as many elements of U and at most l holes.
Now we pass to the other direction. It is enough to show the condition (ii) of the marriage theorem: For every set $X \subset D$ we have $|F(X)| \geq |X|$. Indeed, the theorem claims that then we can "marry" the elements of D with the elements of U, which means exactly covering T by diamonds. So, assume to the contrary that $|F(X)| < |X|$ for some set X. Note that two elements of D having a common neighbor must share a vertex; this means that we can focus on connected sets X. Consider an upward triangle of side 3. It contains three elements of D; if two of them are in X, adding the third one to X increases $F(X)$ by at most 1, so $|F(X)| < |X|$ still holds. Continuing this procedure we will end up with a set X forming an upward sub-triangle of T and satisfying $|F(X)| < |X|$, which contradicts the conditions of the problem. This contradiction proves that $|F(X)| \geq |X|$ for every set X.

13. Consider a polyhedron $\mathscr{P}$ with v vertices, e edges and f faces. Consider the map σ to the unit sphere S taking each vertex, edge or face x of $\mathscr{P}$ to the set of outward unit normal vectors (i.e. points on S) to the support planes of $\mathscr{P}$ containing x. Thus σ maps faces to points on S, edges to shorter arcs of big circles connecting some pairs of these points, and vertices to spherical regions formed by these arcs. These points, arcs and regions on S form a "spherical polyhedron" $\mathscr{G}$.
We now translate the conditions of the problem into the language of $\mathscr{G}$. Denote by $\overline{x}$ the image of x in reflection in the center of S. No edge of $\mathscr{P}$ being parallel to another edge or face means that the big circle of any edge e of $\mathscr{G}$ does not contain any vertex V non-incident to e. Also note that vertices A and B of $\mathscr{P}$ are antipodal if and only if $\sigma(A)$ and $\overline{\sigma(B)}$ intersect, and that the midpoints of edges a and b are antipodal if and only if $\sigma(a)$ and $\overline{\sigma(b)}$ intersect.
Consider the union $\mathscr{F}$ of $\mathscr{G}$ and $\overline{\mathscr{G}}$. The faces of $\mathscr{F}$ are the intersections of faces of $\mathscr{G}$ and $\overline{\mathscr{G}}$, so their number equals $2A$. Similarly, the edges of $\mathscr{G}$ and $\overline{\mathscr{G}}$ have $2B$ intersections, so $\mathscr{F}$ has $2e + 4B$ edges and $2f + 2B$ vertices. Now Euler's theorem for $\mathscr{F}$ gives us $2e + 4B + 2 = 2A + 2f + 2B$, and therefore $A - B = e - f + 1$.

14. The condition of the problem implies that $\angle PBC + \angle PCB = 90^\circ - \alpha/2$, i.e. $\angle BPC = 90^\circ + \alpha/2 = \angle BIC$. Thus P lies on the circumcircle ω of $\triangle BCI$. It

is well known that the center M of ω is the second intersection of AI with the circumcircle of $\triangle ABC$. Therefore $AP \geq AM - MP = AM - MI = AI$, with equality if and only if $P \equiv I$.

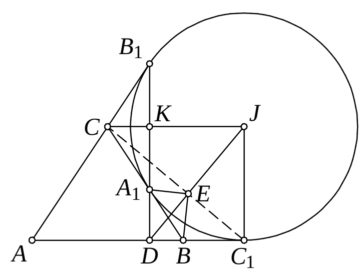
15. The relation $AK/KB = DL/LC$ implies that AD, BC and KL have a common point O. Moreover, since $\angle APB = 180^\circ - \angle ABC$ and $\angle DQC = 180^\circ - \angle BCD$, line BC is tangent to the circles APB and CQD. These two circles are homothetic with respect to O, so if OP meets circle APB again at P', we have $\angle PQC = \angle PP'B = \angle PBC$, showing that P, Q, B, C lie on a circle.

16. Let the diagonals AC and BD meet at Q and AD and CE meet at R. The quadrilaterals $ABCD$ and $ACDE$ are similar, so $AQ/QC = AR/RD$. Now if AP meets CD at M, the Ceva theorem gives us $\frac{CM}{MD} = \frac{CQ}{QA} \cdot \frac{AR}{RD} = 1$.

17. Let M be the point on AC such that $JM \parallel KL$. It is enough to prove that $AM = 2AL$.
From $\angle BDA = \alpha$ we obtain that $\angle JDM = 90^\circ - \frac{\alpha}{2} = \angle KLA = \angle JMD$; hence $JM = JD$ and the tangency point of the incircle of $\triangle BCD$ with CD is the midpoint T of segment MD. Therefore, $DM = 2DT = BD + CD - BC = AB - BC + CD$, which gives us

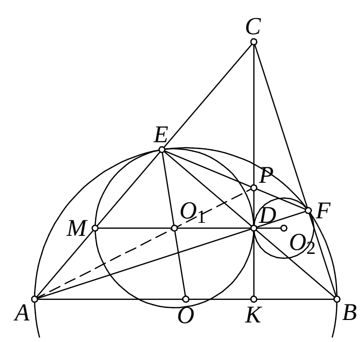
$$AM = AD + DM = AC + AB - BC = 2AL.$$

18. Let A_1B_1 and CJ intersect at K. Then JK is parallel and equal to C_1D and $DC_1/C_1J = JK/JB_1 = JB_1/JC = C_1J/JC$, so the right triangles DC_1J and C_1JC are similar; hence $C_1C \perp DJ$. Thus E belongs to CC_1. Now the points A_1, B_1 and E lie on the circle with diameter CJ. Therefore $\angle DBA_1 = \angle A_1CJ = \angle A_1ED$, implying that BEA_1D is cyclic; hence $\angle A_1EB = 90^\circ$. Likewise, $ADEB_1$ is cyclic because $\angle EB_1A = \angle EJC = \angle EDC_1$, so $\angle AEB_1 = 90^\circ$.



Second solution. The segments JA_1, JB_1, JC_1 are tangent to the circles with diameters A_1B, AB_1, C_1D. Since $JA_1^2 = JB_1^2 = JC_1^2 = JD \cdot JE$, E lies on the first two circles (with diameters A_1B and AB_1), so $\angle AEB_1 = \angle A_1EB = 90^\circ$.

19. The homothety with center E mapping ω_1 to ω maps D to B, so D lies on BE; analogously, D lies on AF. Let AE and BF meet at point C. The lines BE and AF are the altitudes of triangle ABC, so D is the orthocenter and C lies on t. Let the line through D parallel to AB meet AC at M. The centers O_1 and O_2 are the midpoints of DM and DN respectively.



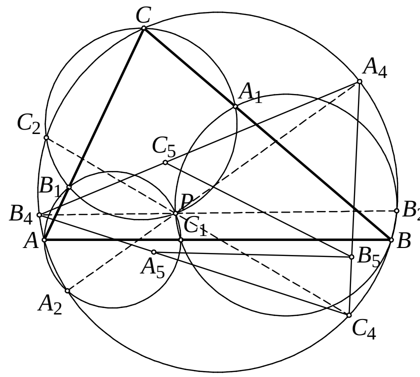
We have thus reduced the problem to a classical triangle geometry problem: If CD and EF intersect at P, we should prove that points A, O_1 and P are collinear (analogously, so are B, O_2, P). By the Menelaus theorem for triangle CDM, this is equivalent to $\frac{CA}{AM} = \frac{CP}{PD}$, which is again equivalent to $\frac{CK}{KD} = \frac{CP}{PD}$ (because $DM \parallel AB$), where K is the foot of the altitude from C to AB. The last equality immediately follows from the fact that the pairs C, D; P, K are harmonically adjoint.

20. Let I be the incenter of $\triangle ABC$. It is well known that T_aT_c and T_aT_b are the perpendicular bisectors of the segments BI and CI respectively. Let T_aT_b meet AC at P and ω_b at U, and let T_aT_c meet AB at Q and ω_c at V. We have $\angle BIQ = \angle IBQ = \angle IBC$, so $IQ \parallel BC$; similarly $IP \parallel BC$. Hence PQ is the line through I parallel to BC.
The homothety from T_b mapping ω_b to the circumcircle ω of ABC maps the tangent t to ω_b at U to the tangent to ω at T_a which is parallel to BC. It follows that $t \parallel BC$. Let t meet AC at X. Since $XU = XM_b$ and $\angle PUM_b = 90^\circ$, X is the midpoint of PM_b. Similarly, the tangent to ω_c at V meets QM_c at its midpoint Y. But since $XY \parallel PQ \parallel M_bM_c$, points U, X, Y, V are collinear, so t coincides with the common tangent p_a. Thus p_a runs midway between I and M_bM_c. Analogous conclusions hold for p_b and p_c, so these three lines form a triangle homothetic to the triangle $M_aM_bM_c$ from center I in ratio $\frac{1}{2}$ which is therefore similar to the triangle ABC in ratio $\frac{1}{4}$.

21. The following proposition is easy to prove:
Lemma. For an arbitrary point X inside a convex quadrilateral $ABCD$, circles ADX and BCX are tangent at X if and only if $\angle ADX + \angle BCX = \angle AXB$.
Let Q be the second intersection point of the circles ABP and CDP (we assume $Q \not\equiv P$; the opposite case is similarly handled). It follows from the conditions of the problem that Q lies inside quadrilateral $ABCD$ (since $\angle BCP + \angle BAP < 180^\circ$, C is outside the circumcircle of APB; the same holds for D). If Q is inside $\triangle APD$ (the other case is similar), we have $\angle BQC = \angle BQP + \angle PQC = \angle BAP + \angle CDP \le 90^\circ$. Similarly $\angle AQD \le 90^\circ$. Moreover, $\angle ADQ + \angle BCQ = \angle ADP + \angle BCP = \angle APB = \angle AQB$ implies that circles ADQ and BCQ are tangent at Q. Therefore the interiors of the semicircles with diameters AD and BC are disjoint and if M, N are the midpoints of AD and BC respectively, we have $2MN \ge AD + BC$. On the other hand, $2MN \le AB + CD$ because $\overrightarrow{BA} + \overrightarrow{CD} = 2\overrightarrow{MN}$, and the statement of the problem immediately follows.

22. We work with oriented angles modulo 180°. For two lines a, b we denote by $\angle(l, m)$ the angle of counterclockwise rotation transforming a to b; also, by $\angle ABC$ we mean $\angle(BA, BC)$.
It is well-known that the circles AB_1C_1, BC_1A_1 and CA_1B_1 have a common point, say P. Let O be the circumcenter of ABC. Denote $\angle PB_1C = \angle PC_1A = \angle PA_1B = \varphi$. Let A_2P, B_2P, C_2P meet the circle ABC again at A_4, B_4, C_4, respectively. Since $\angle A_4A_2A = \angle PA_2A = \angle PC_1A = \varphi$ and thus $\angle A_4OA = 2\varphi$ etc, $\triangle ABC$ is the image of $\triangle A_4B_4C_4$ under rotation $\mathscr{R}$ about O by 2φ.

Therefore $\angle(AB_4, PC_1) = \angle B_4AB + \angle AC_1P = \varphi - \varphi = 0$, so $AB_4 \parallel PC_1$. Let PC_1 intersect A_4B_4 at C_5; define A_5, B_5 analogously. Then $\angle B_4C_5P = \angle A_4B_4A = \varphi$, so $AB_4C_5C_1$ is an isosceles trapezoid with $BC_3 = AC_1 = B_4C_5$. Similarly, $AC_3 = A_4C_5$, so C_3 is the image of C_5 under $\mathscr{R}$; similar statements hold for A_5, B_5. Thus $\triangle A_3B_3C_3 \cong \triangle A_5B_5C_5$. It remains to show that $\triangle A_5B_5C_5 \sim \triangle A_2B_2C_2$.



We have seen that $\angle A_4B_5P = \angle B_4C_5P$, which implies that P lies on the circle $A_4B_5C_5$. Analogously, P lies on the circle $C_4A_5B_5$. Therefore

$$\begin{aligned}\angle A_2B_2C_2 &= \angle A_2B_2B_4 + \angle B_4B_2C_2 = \angle A_2A_4B_4 + \angle B_4C_4C_2 \\ &= \angle PA_4C_5 + \angle A_5C_4P = \angle PB_5C_5 + \angle A_5B_5P = \angle A_5B_5C_5,\end{aligned}$$

and similarly for the other angles, which is what we wanted.

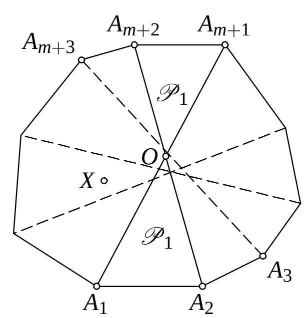
23. Let S_i be the area assigned to side A_iA_{i+1} of polygon $\mathscr{P} = A_1 \dots A_n$ of area S. We start with the following auxiliary statement.

Lemma. At least one of the areas $S_1, \dots, S_n$ is not smaller than $2S/n$.

Proof. It suffices to show the statement for even n. The case of an odd n will then follow immediately from this case applied on the degenerated $2n$-gon $A_1A_1' \dots A_nA_n'$, where A_i' is the midpoint of A_iA_{i+1}.

Let $n = 2m$. For $i = 1, 2, \dots, m$, denote by T_i the area of the region $\mathscr{P}_i$ inside the polygon bounded by the diagonals A_iA_{m+i}, $A_{i+1}A_{m+i+1}$ and the sides A_iA_{i+1}, $A_{m+i}A_{m+i+1}$. We observe that the regions $\mathscr{P}_i$ cover the entire polygon. Indeed, let X be an arbitrary point inside the polygon, to the left (without loss of generality) of the ray A_1A_{m+1}.

Then X is to the right of the ray $A_{m+1}A_1$, so there is a k such that X is to the left of ray A_kA_{k+m} and to the right of ray $A_{k+1}A_{k+m+1}$, i.e. $X \in \mathscr{P}_k$. It follows that $T_1 + \dots + T_m \geq S$; hence at least one T_i is not smaller than $2S/n$, say $T_1 \geq 2S/n$.



Let O be the intersection point of A_1A_{m+1} and A_2A_{m+2}, and let us assume without loss of generality that $S_{A_1A_2O} \geq S_{A_{m+1}A_{m+2}O}$ and $A_1O \geq OA_{m+1}$. Then we have $S_1 \geq S_{A_1A_2A_{m+2}} = S_{A_1A_2O} + S_{A_1A_{m+2}O} \geq S_{A_1A_2O} + S_{A_{m+1}A_{m+2}O} = T_1 \geq 2S/n$, which proves the lemma.

If, contrary to the assertion, $\sum \frac{S_i}{S} < 2$, we can choose rational numbers $q_i = 2m_i/N$ with $N = m_1 + \cdots + m_n$ such that $q_i > S_i/S$. However, considering the given polygon as a degenerated N-gon obtained by division of side A_iA_{i+1} into m_i equal parts for each i and applying the lemma we obtain $S_i/m_i \geq 2S/N$, i.e. $S_i/S \geq q_i$ for some i, a contradiction.

Equality holds if and only if $\mathscr{P}$ is centrally symmetric.

Second solution. We say that vertex V is assigned to side a of a convex (possibly degenerate) polygon $\mathscr{P}$ if the triangle determined by a and V has the maximum area S_a among the triangles with side a contained in $\mathscr{P}$. Denote $\sigma(\mathscr{P}) = \sum_a S_a$ and $\delta(\mathscr{P}) = \sigma(\mathscr{P}) - 2[\mathscr{P}]$. We use induction on the number n of pairwise non-parallel sides of $\mathscr{P}$ to show that $\delta(\mathscr{P}) \geq 0$ for every polygon $\mathscr{P}$. This is obviously true for $n = 2$, so let $n \geq 3$.

There exist two adjacent sides AB and BC whose respective assigned vertices U and V are distinct. Let the lines through U and V parallel to AB and BC respectively intersect at point X. Assume w.l.o.g. that there are no sides of $\mathscr{P}$ lying on UX and VX. Call the sides and vertices of $\mathscr{P}$ lying within the triangle UVX *passive* (excluding vertices U and V). It is easy to see that no passive vertex is assigned to any side of $\mathscr{P}$ and that vertex B is assigned to every passive side. Now replace all passive vertices of $\mathscr{P}$ by X, obtaining a polygon $\mathscr{P}'$. Vertex B is assigned to sides UX and VX or $\mathscr{P}'$, so the sum of areas assigned to passive sides increases by the area S of the part of quadrilateral $BUXV$ lying outside $\mathscr{P}$; the other assigned areas do not change. Thus σ increases by S. On the other hand, the area of the polygon also increases by S, so δ decreases by S.

Note that the change from $\mathscr{P}$ to $\mathscr{P}'$ decreases the number of nonparallel sides. Thus by the inductive hypothesis we have $\delta(\mathscr{P}) \geq \delta(\mathscr{P}') \geq 0$.

Third solution. To each convex n-gon $\mathscr{P} = A_1A_2\dots A_n$ we assign a centrally symmetric $2n$-gon $\mathscr{Q}$, called the *associate* of $\mathscr{P}$, as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $b_1, \cdots, b_{2n}$ counterclockwise so that $b_{n+i} = -b_i$ for $1 \leq i \leq n$. Then take $\mathscr{Q}$ to be the polygon $B_1B_2\dots B_{2n}$ with $\overrightarrow{B_iB_{i+1}} = b_i$. Denote by a_i the side of $\mathscr{P}$ corresponding to b_i ($i = 1, \dots, n$).

The distance between the parallel sides B_iB_{i+1} and $B_{n+i}B_{n+i+1}$ of $\mathscr{Q}$ equals twice the maximum height of $\mathscr{P}$ to the side a_i. Thus, if O is the center of $\mathscr{Q}$, the area of $\triangle B_iB_{i+1}O$ ($i = 1, \dots, n$) is exactly the area S_i assigned to side a_i of $\mathscr{P}$; therefore $[\mathscr{Q}] = 2\sum S_i$. It remains to show that $d(\mathscr{P}) = [\mathscr{Q}] - 4[\mathscr{P}] \geq 0$.

(i) Suppose that $\mathscr{P}$ has two parallel sides a_i and a_j, where $a_j \geq a_i$ and remove from it the parallelogram D determined by a_i and a part of side a_j. We obtain a polygon $\mathscr{P}'$ with a smaller number of nonparallel sides. Then the associate of $\mathscr{P}'$ is obtained from $\mathscr{Q}$ by removing a parallelogram similar to D in ratio 2 (and with area four times that of D); thus $d(\mathscr{P}') = d(\mathscr{P})$.

(ii) Suppose that there is a side b_i $(i \leq n)$ of $\mathscr{Q}$ such that the sum of the angles at its endpoints is greater than 180°. Extend the pairs of sides adjacent to b_i and b_{n+i} to their intersections U and V, thus enlarging $\mathscr{Q}$ by two congruent triangles to a polygon $\mathscr{Q}'$. Then $\mathscr{Q}'$ is the associate of the polygon $\mathscr{P}'$ obtained from $\mathscr{P}$ by attaching a triangle congruent to $B_iB_{i+1}U$ to the side a_i.

Therefore $d(\mathscr{P}')$ equals $d(\mathscr{P})$ minus twice the area of the attached triangle.

By repeatedly performing the operations (i) and (ii) to polygon $\mathscr{P}$ we will eventually reduce it to a parallelogram E, thereby decreasing the value of d. Since $d(E) = 0$, it follows that $d(\mathscr{P}) \geq 0$.

Remark. Polygon $\mathscr{Q}$ is the Minkowski sum of $\mathscr{P}$ and a polygon centrally symmetric to $\mathscr{P}$. Thus the inequality $[\mathscr{Q}] \geq 4[\mathscr{P}]$ is a direct consequence of the Brunn-Minkowski inequality.

24. Obviously $x \geq 0$. For $x = 0$ the only solutions are $(0, \pm 2)$. Now let (x, y) be a solution with $x > 0$. Assume w.l.o.g. that $y > 0$. The equation rewritten as $2^x(1 + 2^{x+1}) = (y-1)(y+1)$ shows that one of the factors $y \pm 1$ is divisible by 2 but not by 4 and the other by 2^{x-1} but not by 2^x; hence $x \geq 3$. Thus $y = 2^{x-1}m + \varepsilon$, where m is odd and $\varepsilon = \pm 1$. Plugging this in the original equation and simplifying yields
$$2^{x-2}(m^2 - 8) = 1 - \varepsilon m. \qquad (*)$$
As $m = 1$ is obviously impossible, we have $m \geq 3$ and hence $\varepsilon = -1$. Now $(*)$ gives us $2(m^2 - 8) \leq 1 + m$, implying $m = 3$ which leads to $x = 4$ and $y = 23$. Thus all solutions are $(0, \pm 2)$ and $(4, \pm 23)$.

25. If x is rational, its digits repeat periodically starting at some point. If n is the length of the period of x, the sequence $2, 2^2, 2^3, \dots$ is eventually periodic modulo n, so the corresponding digits of x (i.e. the digits of y) also make an eventually periodic sequence, implying that y is rational.

26. Consider $g(n) = [\frac{n}{1}] + [\frac{n}{2}] + \cdots + [\frac{n}{n}] = nf(n)$ and define $g(0) = 0$. Since for any k the difference $[\frac{n}{k}] - [\frac{n-1}{k}]$ equals 1 if k divides n and 0 otherwise, we obtain that $g(n) = g(n-1) + d(n)$, where $d(n)$ is the number of positive divisors of n. Thus $g(n) = d(1) + d(2) + \cdots + d(n)$ and $f(n)$ is the arithmetic mean of the numbers $d(1), \dots, d(n)$. Therefore, (a) and (b) will follow if we show that each of $d(n+1) > f(n)$ and $d(n+1) < f(n)$ holds infinitely often. But $d(n+1) < f(n)$ holds whenever $n+1$ is prime, and $d(n+1) > f(n)$ holds whenever $d(n+1) > d(1), \dots, d(n)$ (which clearly holds for infinitely many n).

27. We first show that every fixed point x of Q is in fact a fixed point of $P \circ P$. Consider the sequence given by $x_0 = x$ and $x_{i+1} = P(x_i)$ for $i \geq 0$. Assume $x_k = x_0$. We know that $u - v$ divides $P(u) - P(v)$ for every two distinct integers u, v. In particular,
$$d_i = x_{i+1} - x_i \mid P(x_{i+1}) - P(x_i) = x_{i+2} - x_{i+1} = d_{i+1}$$
for all i, which together with $d_k = d_0$ implies $|d_0| = |d_1| = \cdots = |d_k|$. Suppose that $d_1 = d_0 = d \neq 0$. Then $d_2 = d$ (otherwise $x_3 = x_1$ and x_0 will never occur in the sequence again). Similarly, $d_3 = d$ etc, and hence $x_i = x_0 + id \neq x_0$ for all i, a contradiction. It follows that $d_1 = -d_0$, so $x_2 = x_0$ as claimed. Thus we can assume that $Q = P \circ P$.

If every integer t with $P(P(t)) = t$ also satisfies $P(t) = t$, the number of solutions is clearly at most $\deg P = n$. Suppose that $P(t_1) = t_2$, $P(t_2) = t_1$, $P(t_3) = t_4$ i

$P(t_4) = t_3$, where $t_1 \neq t_{2,3,4}$ (but not necessarily $t_3 \neq t_4$). Since $t_1 - t_3$ divides $t_2 - t_4$ and vice versa, we conclude that $t_1 - t_3 = \pm(t_2 - t_4)$. Assume that $t_1 - t_3 = t_2 - t_4$, i.e. $t_1 - t_2 = t_3 - t_4 = u \neq 0$. Since the relation $t_1 - t_4 = \pm(t_2 - t_3)$ similarly holds, we obtain $t_1 - t_3 + u = \pm(t_1 - t_3 - u)$ which is impossible. Therefore, we must have $t_1 - t_3 = t_4 - t_2$, which gives us $P(t_1) + t_1 = P(t_3) + t_3 = c$ for some c. It follows that all integral solutions t of the equation $P(P(t)) = t$ satisfy $P(t) + t = c$, and hence their number does not exceed n.

28. Every prime divisor p of $\frac{x^7-1}{x-1} = x^6 + \cdots + x + 1$ is congruent to 0 or 1 modulo 7. Indeed, If $p \mid x - 1$, then $\frac{x^7-1}{x-1} \equiv 1 + \cdots + 1 \equiv 7 \pmod p$, so $p = 7$; otherwise the order of x modulo p is 7 and hence $p \equiv 1 \pmod 7$. Therefore every positive divisor d of $\frac{x^7-1}{x-1}$ satisfies $d \equiv 0$ or $1 \pmod 7$.
Now suppose (x, y) is a solution of the given equation. Since $y - 1$ and $y^4 + y^3 + y^2 + y + 1$ divide $\frac{x^7-1}{x-1} = y^5 - 1$, we have $y \equiv 1$ or 2 and $y^4 + y^3 + y^2 + y + 1 \equiv 0$ or $1 \pmod 7$. However, $y \equiv 1$ or 2 implies that $y^4 + y^3 + y^2 + y + 1 \equiv 5$ or $3 \pmod 7$, which is impossible.

29. All representations of n in the form $ax + by$ $(x, y \in \mathbb{Z})$ are given by $(x, y) = (x_0 + bt, y_0 - at)$, where x_0, y_0 are fixed and $t \in \mathbb{Z}$ is arbitrary. The following lemma enables us to determine $w(n)$.
Lemma. The equality $w(ax + by) = |x| + |y|$ holds if and only if:
  (i) $\frac{a-b}{2} < y \leq \frac{a+b}{2}$ and $x \geq y - \frac{a+b}{2}$; or
  (ii) $-\frac{a-b}{2} \leq y \leq \frac{a-b}{2}$ and $x \in \mathbb{Z}$; or
  (iii) $-\frac{a+b}{2} \leq y < -\frac{a-b}{2}$ and $x \leq y + \frac{a+b}{2}$.
Proof. Assume w.l.o.g. that $y \geq 0$. We have $w(ax + by) = |x| + y$ if and only if $|x + b| + |y - a| \geq |x| + y$ and $|x - b| + (y + a) \geq |x| + y$, where the latter is obviously true and the former clearly implies $y < a$. Then the former inequality becomes $|x + b| - |x| \geq 2y - a$. We distinguish three cases: if $y \leq \frac{a-b}{2}$ then $2y - a \leq b$ and the previous inequality always holds; for $\frac{a-b}{2} < y \leq \frac{a+b}{2}$ it holds if and only if $x \geq y - \frac{a+b}{2}$; and for $y > \frac{a+b}{2}$ it never holds.
Now let $n = ax + by$ be a local champion with $w(n) = |x| + |y|$. As in lemma, we distinguish three cases:
(i) $\frac{a-b}{2} < y \leq \frac{a+b}{2}$. Then $x + 1 \geq y - \frac{a+b}{2}$ by the lemma, so $w(n + a) = |x + 1| + y$ (because $n + a = a(x + 1) + by$). Since $w(n + a) \leq w(n)$, we must have $x < 0$. Likewise, $w(n - a)$ equals either $|x - 1| + y = w(n) + 1$ or $|x + b - 1| + a - y$. The condition $w(n - a) \leq w(n)$ leads to $x \leq y - \frac{a+b-1}{2}$; hence $x = y - [\frac{a+b}{2}]$ and $w(n) = [\frac{a+b}{2}]$. Now $w(n - b) = -x + y - 1 = w(n) - 1$ and $w(n + b) = (x + b) + (a - 1 - y) = a + b - 1 - [\frac{a+b}{2}] \leq w(n)$, so n is a local champion. Conversely, every $n = ax + by$ with $\frac{a-b}{2} < y \leq \frac{a+b}{2}$ and $x = y - [\frac{a+b}{2}]$ is a local champion. Thus we obtain $b - 1$ local champions which are all distinct.
(ii) $|y| \leq \frac{a-b}{2}$. Now we conclude from the lemma that $w(n - a) = |x - 1| + |y|$ and $w(n + a) = |x + 1| + |y|$, and at least one of these two values exceeds $w(n) = |x| + |y|$. Thus n is not a local champion.

(iii) $-\frac{a+b}{2} \le y < -\frac{a-b}{2}$. By taking x, y to $-x, -y$ this case is reduced to case (i), so we again have $b-1$ local champions $n = ax+by$ with $x = y+[\frac{a+b}{2}]$.

It is easy to check that the sets of local champions from cases (i) and (iii) coincide if a and b are both odd (so we have $b-1$ local champions in total), and are otherwise disjoint (then we have $2(b-1)$ local champions).

30. We shall show by induction on n that there exists an arbitrarily large m satisfying $2^m \equiv -m \pmod{n}$. The case $n = 1$ is trivial; assume that $n > 1$.
Recall that the sequence of powers of 2 modulo n is eventually periodic with the period dividing $\varphi(n)$; thus $2^x \equiv 2^y$ whenever $x \equiv y \pmod{\varphi(n)}$ and x and y are large enough. Let us consider m of the form $m \equiv -2^k \pmod{n\varphi(n)}$. Then the congruence $2^m \equiv -m \pmod{n}$ is equivalent to $2^m \equiv 2^k \pmod{n}$, and this holds whenever $-2^k \equiv m \equiv k \pmod{\varphi(n)}$ and m, k are large enough. But the existence of m and k is guartanteed by the inductive hypothesis for $\varphi(n)$, so the induction is complete.

A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathscr{B}(A,B,C)$, $A-B-C$: indicates the relation of *betweenness*, i.e., that B is between A and C (this automatically means that A,B,C are different collinear points).
- $A = l_1 \cap l_2$: indicates that A is the intersection point of the lines l_1 and l_2.
- AB: line through A and B, segment AB, length of segment AB (depending on context).
- $[AB$: ray starting in A and containing B.
- $(AB$: ray starting in A and containing B, but without the point A.
- (AB): open interval AB, set of points between A and B.
- $[AB]$: closed interval AB, segment AB, $(AB) \cup \{A,B\}$.
- $(AB]$: semiopen interval AB, closed at B and open at A, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- ABC: plane determined by points A,B,C, triangle ABC ($\triangle ABC$) (depending on context).
- $[AB,C$: half-plane consisting of line AB and all points in the plane on the same side of AB as C.
- $(AB,C$: $[AB,C$ without the line AB.

- $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$, $\overrightarrow{a} \cdot \overrightarrow{b}$: scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$.
- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle ABC (unless otherwise indicated).
- $k(O, r)$: circle k with center O and radius r.
- $d(A, p)$: distance from point A to line p.
- $S_{A_1A_2\ldots A_n}$, $[A_1A_2\ldots A_n]$: area of n-gon $A_1A_2\ldots A_n$ (special case for $n = 3$, S_{ABC}: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo n, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo p, p being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in x with integer and real coefficients respectively.
- R^*: the set of nonzero elements of a ring R.
- $R[\alpha]$, $R(\alpha)$, where α is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- X_0: $X \cup \{0\}$ for X such that $0 \notin X$.
- X^+, X^-, $aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to x.
- $\lceil x \rceil$: the smallest integer greater than or equal to x.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points AB, the absolute value of the number x, the number of elements of the set S (respectively).
- (x, y), (m, n), (a, b): (ordered) pair x and y, the greatest common divisor of integers m and n, the open interval between real numbers a and b (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).

- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

B

Codes of the Countries of Origin

ARG	Argentina	HKG	Hong Kong	POL	Poland
ARM	Armenia	HUN	Hungary	POR	Portugal
AUS	Australia	ICE	Iceland	PRK	Korea, North
AUT	Austria	INA	Indonesia	PUR	Puerto Rico
BEL	Belgium	IND	India	ROM	Romania
BLR	Belarus	IRE	Ireland	RUS	Russia
BRA	Brazil	IRN	Iran	SAF	South Africa
BUL	Bulgaria	ISR	Israel	SER	Serbia
CAN	Canada	ITA	Italy	SIN	Singapore
CHN	China	JAP	Japan	SLO	Slovenia
COL	Colombia	KAZ	Kazakhstan	SMN	Serbia and Montenegro
CRO	Croatia	KOR	Korea, South	SPA	Spain
CUB	Cuba	KUW	Kuwait	SVK	Slovakia
CYP	Cyprus	LAT	Latvia	SWE	Sweden
CZE	Czech Republic	LIT	Lithuania	THA	Thailand
CZS	Czechoslovakia	LUX	Luxembourg	TUN	Tunisia
EST	Estonia	MCD	Macedonia	TUR	Turkey
FIN	Finland	MEX	Mexico	TWN	Taiwan
FRA	France	MON	Mongolia	UKR	Ukraine
FRG	Germany, FR	MOR	Morocco	USA	United States
GBR	United Kingdom	NET	Netherlands	USS	Soviet Union
GDR	Germany, DR	NOR	Norway	UZB	Uzbekistan
GEO	Georgia	NZL	New Zealand	VIE	Vietnam
GER	Germany	PER	Peru	YUG	Yugoslavia
GRE	Greece	PHI	Philippines		