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IMO Shortlist 2007

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Problems

1.1 The Forty-Eighth IMO
Hanoi, Vietnam, July 19–31, 2007

1.1.1 Contest Problems

First Day (July 25)

1. Real numbers $a_1, a_2, \dots, a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j \mid 1 \le j \le i\} - \min\{a_j \mid i \le j \le n\}$$

and let $d = \max\{d_i \mid 1 \le i \le n\}$.
(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| \mid 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that equality holds in $(*)$.

2. Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

3. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.
Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Second Day (July 26)

4. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.
5. Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.
6. Let n be a positive integer. Consider
$$S=\{(x,y,z) \mid x,y,z\in\{0,1,\dots,n\},\ x+y+z>0\}$$
as a set of $(n+1)^3-1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0,0,0)$.

1.1.2 Shortlisted Problems

1. **A1 (NZL)** IMO1 Given a sequence $a_1,a_2,\dots,a_n$ of real numbers, for each i $(1\le i\le n)$ define
$$d_i=\max\{a_j : 1\le j\le i\}-\min\{a_j : i\le j\le n\}$$
and let $d=\max\{d_i : 1\le i\le n\}$.
(a) Prove that for arbitrary real numbers $x_1\le x_2\le\cdots\le x_n$,
$$\max\{|x_i-a_i| : 1\le i\le n\}\ge \frac{d}{2}. \tag{1}$$
(b) Show that there exists a sequence $x_1\le x_2\le\cdots\le x_n$ of real numbers such that we have equality in (1).
2. **A2 (BUL)** Consider those functions $f:\mathbb{N}\to\mathbb{N}$ which satisfy the condition
$$f(m+n)\ge f(m)+f(f(n))-1, \text{ for all } m,n\in\mathbb{N}.$$
Find all possible values of $f(2007)$.
3. **A3 (EST)** Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that
$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}.$$
4. **A4 (THA)** Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that
$$f(x+f(y))=f(x+y)+f(y)$$
for all $x,y\in\mathbb{R}^+$.

5. **A5 (CRO)** Let $c > 2$, and let $a(1), a(2), \dots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \le 2a(m) + 2a(n) \text{ for all } m,n \ge 1, \text{ and}$$
$$a(2^k) \le \frac{1}{(k+1)^c} \text{ for all } k \ge 0.$$

Prove that the sequence $a(n)$ is bounded.

6. **A6 (POL)** Let $a_1, a_2, \dots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \cdots + a_{100}^2 = 1$. Prove that

$$a_1^2 a_2 + a_2^2 a_3 + \cdots + a_{100}^2 a_1 < \frac{12}{25}.$$

7. **A7 (NET)**[IMO6] Let $n > 1$ be an integer. Consider the following subset of the space:

$$S = \{(x,y,z) | x,y,z \in \{0,1,\dots,n\}, x+y+z > 0\}.$$

Find the smallest number of planes that jointly contain all $(n+1)^3 - 1$ points of S but none of them passes through the origin.

8. **C1 (SER)** Let n be an integer. Find all sequences $a_1, a_2, \dots, a_{n^2+n}$ satisfying the following conditions:
   (i) $a_i \in \{0,1\}$ for all $1 \le i \le n^2 + n$;
   (ii) $a_{i+1} + a_{i+2} + \cdots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \cdots + a_{i+2n}$ for all $0 \le i \le n^2 - n$.

9. **C2 (JAP)** A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

10. **C3 (NET)** Find all positve integers n, for which the numbers in the set $S = \{1, 2, \dots, n\}$ can be colored red and blue, with the following condition being satisfied: the set $S \times S \times S$ contains exactly 2007 ordered triples (x,y,z) such that
    (i) x,y,z are of the same color and
    (ii) $x + y + z$ is divisible by n.

11. **C4 (IRN)** Let $A_0 = \{a_1, \dots, a_n\}$ be a finite sequence of real numbers. For each $k \ge 0$, from the sequence $A_k = (x_1, \dots, x_n)$ we construct a new sequence A_{k+1} in the following way:
    (i) We choose a partition $\{1, \dots, n\} = I \cup J$, where I and J are two discjoint sets, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

(ii) We set $A_{k+1} = (y_1, \dots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$. Prove that for some k, the sequence A_k contains an element x such that $|x| \geq n/2$.

12. **C5 (ROM)** In the Cartesian coordinate plane define the strip
$$S_n = \{(x,y) : n \leq x < n+1\}$$
for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

13. **C6 (RUS)**$^{\text{IMO3}}$ In a mathematical competition some competitors are friends; friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. The number of members in a clique is called its *size*.
It is known that the largest size of a clique is even. Prove that the competitors can be arranged in two rooms such that the largest size of a clique in one room is the same as the largest size of a clique in the other room.

14. **C7 (AUT)** Let $\alpha < \frac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and p such that $p > \alpha \cdot 2^n$ and for which one can select $2p$ pairwise distinct subsets $S_1, \dots, S_p, T_1, \dots, T_p$ of the set $\{1, 2, \dots, n\}$ such that $S_i \cap T_j \neq \emptyset$ for all $1 \leq i, j \leq p$.

15. **C8 (UKR)** Given a convex n-gon P in the plane, for every three vertices of P, consider the triangle determined by them. Call such a triangle *good* if all its sides are of unit length. Prove that there are not more than $2n/3$ good triangles.

16. **G1 (CZE)** $^{\text{IMO4}}$ In a triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

17. **G2 (CAN)** Given an isosceles triangle ABC, assume that $AB = AC$. The midpoint of the side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

18. **G3 (UKR)** The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and the line CD separates the points P and Q. Prove that $\angle BQP = \angle DAQ$.

19. **G4 (LUX)** $^{\text{IMO2}}$ Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

20. **G5 (GBR)** Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the modpoints of sides BC, CA, AB respectively. Let P be a variable point on the circumcircle.

Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

21. **G6 (USA)** Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of the quadrilateral $A_1B_1C_1D_1$.
Find the smallest positive real number k such that $kS_1 \geq S$ holds for every convex quadrilateral $ABCD$.

22. **G7 (IRN)** Given an acute triangle ABC with angles α, β, and γ at vertices A, B, and C respectively such that $\beta > \gamma$, let I be its incenter, and R the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally lines DI and KI meet sides AC and BC at E and F respectively. Prove that if $IE = IF$ then $\beta \leq 3\gamma$.

23. **G8 (POL)** A point P lies on the side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of the triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let the lines AC and BD meet at E, and let the lines AK and BL meet at F. Prove that the points E, I, and F are colinear.

24. **N1 (AUT)** Find all pairs (k,n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

25. **N2 (CAN)** Let $b,n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

26. **N3 (NET)** Let X be a set of 10000 integers, none of which is divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a - b + c - d + e$ is not divisible by 47 for any $a,b,c,d,e \in Y$.

27. **N4 (POL)** For every integer $k \geq 2$, prove that 2^{3k} divides the number
$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}}$$
but 2^{3k+1} does not.

28. **N5 (IRN)** Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m,n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

29. **N6 (GBR)** [IMO5] Let k be a positive integer. Prove that the number $(4k^2 - 1)^2$ has a positive divisor of the form $8kn - 1$ if and only if k is even.

30. **N7 (IND)** For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set $\{p_1, \ldots, p_k\}$ of primes, show that there are infinitely many positive integers n such that $d | \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

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Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2007

1. (a) Assume that $d = d_m$ for some index m, and let k and l be the indeces such that $k \le m \le l$ such that $d_m = a_k - a_l$. Then $d_m = a_k - a_l \le (a_k - x_k) + (x_l - a_l)$ hence $a_k - x_k \ge d/2$ or $x_l - a_l \ge d/2$. The claim follows immediately.

   (b) Let $M_i = \max\{a_j : 1 \le j \le i\}$ and $m_i = \min\{a_j : i \le j \le n\}$. Set $x_i = \frac{m_i + M_i}{2}$. Clearly, $m_i \le a_i \le M_i$ and both (m_i) and (M_i) are non-decreasing. Furthermore, $-\frac{d_i}{2} = \frac{m_i - M_i}{2} = x_i - M_i \le x_i - a_i$. Similarly $x_i - a_i \le \frac{d_i}{2}$, hence $\max\{|x_i - a_i| : 1 \le i \le n\} \le \max\left\{\frac{d_i}{2}, 1 \le i \le n\right\}$. Thus, the equality holds in (1) for the sequence $\{x_i\}$.

2. Placing $n = 1$ we get $f(m+1) \ge f(m) + f(f(1)) - 1 \ge f(m)$ hence the function is non-decreasing. Let n_0 be the smallest integer such that $f(n_0) > 1$. If $f(n) = n + k$ for some $k, n \ge 1$ then placing $m = 1$ gives that $f(f(n)) = f(n+k) \ge f(k) + f(f(n)) - 1$ which implies $f(k) = 1$. We immediately get $k < n_0$. Choose maximal k_0 such that there exists $n \in \mathbb{N}$ for which $f(n) = n + k_0$. Then we have $2n + k_0 \ge f(2n) \ge f(n) + f(f(n)) - 1 = n + k_0 + f(n + k_0) - 1 \ge n + k_0 + f(n) - 1 = 2n + (2k_0 - 1)$ hence $2k_0 - 1 \le k_0$, or $k_0 \le 1$. Therefore $f(n) \le n + 1$ and $f(2007) \le 2008$.
   Now we will prove that $f(2007)$ can be any of the numbers $1, 2, \ldots, 2008$. Define the functions

$$f_j(n) = \begin{cases} 1, & n \le 2007 - j, \\ n + j - 2007, & \text{otherwise.} \end{cases}, \ j \le 2007, \text{ and}$$

$$f_{2008}(n) = \begin{cases} n, & 2007 \nmid n, \\ n+1, & 2007 \mid n. \end{cases}$$

   It is easy to verify that f_j satisfy the conditions of the problem for $j = 1, 2, \ldots, 2008$.

3. The inequality $\frac{1+t^2}{1+t^4} < \frac{1}{t}$ holds for all $t \in (0,1)$ because it is equivalent to $0 < t^4 - t^3 - t + 1 = (1-t)(1-t^3)$. Applying it to $t = x^k$ and summing over $k = 1, \ldots, n$ we get $\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}} < \sum_{k=1}^{n} \frac{1}{x^k} = \frac{x^n - 1}{x^n(x-1)} = \frac{y^n}{x^n(1-x)}$. Writing the same relation for y and multiplying by this one gives the desired inequality.

4. Notice that $f(x) > x$ for all x. Indeed, $f(x + f(y)) \ne f(x+y)$ and if $f(y) < y$ for some y, setting $x = y - f(y)$ yields to a contradiction.
   Now we will prove that $f(x) - x$ is injective. If we assume that $f(x) - x = f(y) - y$ for some $x \ne y$ we would have $x + f(y) = y + f(x)$ hence $f(x+y) + f(y) = f(x+y) + f(x)$ implying $f(x) = f(y)$, which is impossible. From the functional equation we conclude that $f(f(x) + f(y)) - (f(x) + f(y)) = f(x+y)$, hence $f(x) + f(y) = f(x') + f(y')$ whenever $x + y = x' + y'$. In particular, we have $f(x) + f(y) = 2f(\frac{x+y}{2})$.
   Our next goal is to prove that f is injective. If $f(x) = f(x+h)$ for some $h > 0$ then $f(x) + f(x+2h) = 2f(x+h) = 2f(x)$ hence $f(x) = f(x+2h)$, and by induction

$f(x+nh) = f(x)$. Therefore, $0 < f(x+nh) - (x+nh) = f(x) - x - nh$ for every n, which is impossible.

We now have $f(f(x)+f(y)) = f(f(x)+y) + f(y) = 2f(\frac{f(x)}{2}+y)$ and by symmetry $f(f(x)+f(y)) = 2f(\frac{f(y)}{2}+x)$. Hence $\frac{f(x)}{2}+y = \frac{f(y)}{2}+x$, thus $f(x)-2x = c$ for some $c \in \mathbb{R}$. The functional equation forces $c = 0$. It is easy to verify that $f(x) = 2x$ satisfies the given relation.

5. Defining $a(0)$ to be 0 the relations in the problem remain to hold. It follows by induction that $a(n_1+n_2+\cdots+n_k) \le 2a(n_1) + 2^2 a(n_2) + \cdots + 2^k a(n_k)$. We also have $a(n_1+n_2+\cdots+n_{2^i}) \le 2a(n_1+\cdots+n_{2^{i-1}}) + 2a(n_{2^{i-1}+1}+\cdots+n_{2^i}) \le \cdots \le 2^i(a(n_1)+\cdots+a(n_{2^i}))$. For integer $k \in [2^i, 2^{i+1})$ we have

$$\begin{aligned} a(n_1+\cdots+n_k) &= a(n_1+\cdots+n_k+\underbrace{0+\cdots+0}_{2^{i+1}-k}) \\ &\le 2^{i+1}(a(n_1)+\cdots+a(n_k)+(2^{i+1}-k)a(0)) \\ &\le 2k(a(n_1)+\cdots+a(n_k)). \end{aligned}$$

Assume now that $N \in \mathbb{N}$ is given and let $N = \sum_{i=0}^{K} b_i 2^i$ be its binary representation ($K \in \mathbb{N}$, $b_i \in \{0,1\}$). For each increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of integers we have

$$\begin{aligned} a(N) &= a\left(\sum_n \sum_{i=\tau_{n-1}}^{\max\{\tau_n,K\}} b_i 2^i\right) \le \sum_n 2^n a\left(\sum_{i=\tau_{n-1}}^{\max\{\tau_n,K\}} b_i 2^i\right) \\ &\le \sum_n 2^n \cdot 2(\tau_n - \tau_{n-1}+1) \sum_{i=\tau_{n-1}}^{\max\{\tau_n,K\}} a(2^i) \\ &\le \sum_n (\tau_n - \tau_{n-1}+1)^2 \frac{2^{n+1}}{(\tau_{n-1}+1)^c} \le \sum_n \frac{2^{n+1}\left(\frac{\tau_n+1}{\tau_{n-1}+1}\right)^2}{(\tau_{n-1}+1)^{c-2}}. \end{aligned}$$

Choosing $\tau_n = 2^{\alpha n} - 1$ we get $a(N) \le 2^{2\alpha} \cdot 2^2 \sum_n 2^{n-1-\alpha(c-2)(n-1)}$. Thus, choosing any $\alpha > \frac{1}{c-2}$ would give us the sequence τ_n for which the last series is bounded, which proves the required statement.

6. Using the Cauchy-Schwarz inequality we can bound the left-hand side in the following way: $\frac{1}{3}[a_1(a_{100}^2+2a_1a_2) + a_2(a_1^2+2a_2a_3) + \cdots + a_{100}(a_{99}^2+2a_{100}a_1)] \le \frac{1}{3}\left(a_1^2+\cdots+a_{100}^2\right)^{1/2} \cdot \left(\sum_{k=1}^{100}(a_k^2+2a_{k+1}a_{k+2})\right)^{1/2}$ (the indeces are modulo 100). It suffices to show

$$\sum_{k=1}^{100}(a_k^2+2a_{k+1}a_{k+2})^2 \le 2.$$

Each term of the last sum can be seen as $a_k^4 + 4a_{k+1}^2a_{k+2}^2 + 4a_k^2(a_{k+1}\cdot a_{k+2}) \le (a_k^4 + 2a_k^2a_{k+1}^2 + 2a_k^2a_{k+2}^2) + 4a_{k+1}^2a_{k+2}^2$. The required inequality now follows from $\sum_{k=1}^{100}(a_k^4+2a_k^2a_{k+1}^2+2a_k^2a_{k+2}^2) \le (a_1^2+\cdots+a_{100}^2)^2 = 1$, and $\sum_{k=1}^{100} a_k^2 a_{k+1}^2 \le (a_1^2+a_3^2+\cdots+a_{99}^2)\cdot(a_2^2+a_4^2+\cdots+a_{100}^2) \le \frac{1}{4}\left(a_1^2+\cdots+a_{100}^2\right)^2 = \frac{1}{4}$.

7. The union of the planes $x = i$, $y = i$, and $z = i$ for $1 \le i \le n$ contains S and doesn't contain 0. Assume now that there exists a collection $\{a_i x + b_i y + c_i z + d_i = 0 : 1 \le i \le N\}$ of $N < 3n$ planes with the described properties. Consider the polynomial $P(x,y,z) = \prod_{i=1}^{N}(a_i x + b_i y + c_i z + d_i)$.
   Let $\delta_0 = 1$, and choose the numbers $\delta_1, \dots, \delta_n$ such that $\sum_{i=0}^{n} \delta_i i^m = 0$ for $m = 0, 1, 2, \dots, n-1$ (here we assume that $0^0 = 1$). The choice of these numbers is possible because the given linear system in $(\delta_1, \dots, \delta_n)$ has the Vandermonde determinant.
   Let $S = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \delta_i \delta_j \delta_k P(i,j,k)$. By the construction of P we know that $P(0,0,0) \neq 0$ and $P(i,j,k) = 0$ for all other choices of $i,j,k \in \{0,1,\dots,n\}$. Therefore $S = \delta_0^3 P(0,0,0)$. On the other hand expanding P as $P(x,y,z) = \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} x^\alpha y^\beta z^\gamma$ we get:
$$S = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \delta_i \delta_j \delta_k \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} i^\alpha j^\beta k^\gamma$$
$$= \sum_{\alpha+\beta+\gamma \le N} p_{\alpha,\beta,\gamma} \left(\sum_{i=0}^{n} \delta_i i^\alpha\right)\left(\sum_{j=0}^{n} \delta_j j^\beta\right)\left(\sum_{k=0}^{n} \delta_k k^\gamma\right) = 0$$
   because for every choice of α, β, γ at least one of them is less than n making the corresponding sum in the last expression equal to 0. This is a contradiction, hence the required number of planes is $3n$.

8. Let $S_k^m = a_k + a_{k+1} + \cdots + a_m$. Since $S_1^n < S_{n+1}^{2n} < \cdots < S_{n^2+1}^{n^2+n}$ and since each of these $n+1$ numbers belongs to $\{0,1,\dots,n\}$ we have that $S_{in+1}^{(i+1)n} = i$. We immediately get $a_1 = a_2 = \cdots = a_n = 0$ and $a_{n^2+1} = a_{n^2+2} = \cdots = a_{n^2+n} = 1$. For every $0 \le k \le n$, consider the sequence $l_k = (S_{k+1}^{k+n}, S_{k+n+1}^{k+2n}, \dots, S_{k+n^2-n+1}^{k+n^2})$. The sequence is strictly increasing, and its elements are from the set $\{0,1,2,\dots,n\}$. Let m be the number that doesn't appear in l_k, and U_k the total sum of the elements of l_k. Since $a_1 + \cdots + a_{n^2+n} = S_1^n + S_{n+1}^{2n} + \cdots + S_{n^2+1}^{n^2+n} = \frac{n(n+1)}{2} = S_1^k + U_k + S_{k+n^2+1}^{n^2+n} = U_k + n - k$ we get $m = n - k$. Therefore
$$S_{k+sn+1}^{k+(s+1)n} = \begin{cases} s, & \text{if } s < n-k, \\ s+1, & \text{if } s \ge n-k. \end{cases}$$
   Using this we get
$$\begin{aligned} S_{k+sn+1}^{k+(s+1)n} &= S_{k-1+sn+1}^{k-1+(s+1)n} + a_{k+(s+1)n} - a_{k-1+sn+1} \\ &= S_{k+sn}^{k-1+(s+1)n} + a_{k+(s+1)n} - a_{k+sn}, \end{aligned}$$
   hence for $s+k < n$ we have $S_{k+sn+1}^{k+(s+1)n} = S_{k+sn}^{k-1+(s+1)n} = s$, and for $s+k \ge n+1$ we have $S_{k+sn+1}^{k+(s+1)n} = S_{k+sn}^{k-1+(s+1)n} = s+1$. Hence $a_{k+(s+1)n} = a_{k+sn}$ if either $s+k < n$ or $s+k \ge n+1$. If $s+k = n$ then $S_{k+sn}^{k-1+(s+1)n} = s$ while $S_{k+sn+1}k + (s+1)n =$

$s+1$. hence $a_{k+(s+1)n} = 1$ and $a_{k+sn} = 0$. Now, by induction we can easily get that for $1 \le u \le n$ and $0 \le v \le n$:

$$a_{u+vn} = \begin{cases} 0, & \text{if } u+v \le n, \\ 1, & \text{if } u+v > n. \end{cases}$$

It is easy to verify that the above sequence satisfies the required properties.

9. Assume the contrary. Consider the minimal such dissection of the square $ABCD$ (i.e. the dissection with the smallest number of rectangles). No two rectangles in this minimal dissection can share an edge. Let $AMNP$ be the rectangle contianing the vertex A, and let $UBVW$ be the rectangle containing B. Assume that $MN \le BV$. Let $MXYZ$ be another rectangle containing the point M (this one could be the same as $UBVW$). We can either have $MN > MZ$ or $MZ > MN$. In the first case the rectangle containing the point Z would have to touch the side CD (it can't touch BC because $WU \ge NM > MZ$). The line MN doesn't intersect any of the interiors of the rectangles. Contradiction
If $MZ > MN$ consider the rectangle containing the point N. It can't touch AD because it can't share the entire side with $AMNP$. Hence it has to touch CD and, again, MN would be a line that doesn't intersect any of the interiors. Contradiction.

10. Let $T = \{(x,y,z) \in S \times S \times S : x+y+z \text{ is divisible by } n\}$. For any pair $(x,y) \in S \times S$ there exists unique $z \in \mathbb{S}$ such that $(x,y,z) \in T$, hence $|T| = n^2$. Let $M \subseteq T$ be the set of those triples that have all elements of the same color. Denote by R and B the sets of red and blue numbers and assume that the number r of red numbers is not less than $n/2$. Consider the following function $F : T \setminus M \to R \times B$: If $(x,y,z) \in T \setminus M$, then $F(x,y,z)$ is defined to be one of the pairs (x,y), (y,z), (z,x) that belongs to $R \times B$ (there exists exactly one such pair). For each element $(p,q) \in R \times B$ there is unique $s \in S$ for which $n|p+q+s$. Then $F(p,q,s) = F(s,p,q) = F(q,s,p) = (p,q)$. Hence $|T \setminus M| = 3|R \times B| = 3r(n-r)$ and $|T| = n^2 - 3r(n-r) = n^2 - 3rn + 3r^2$.
It remains to solve $n^2 - 3nr + 3r^2 = 2007$ in the set $\mathbb{N} \times \mathbb{N}$. First of all, $n = 3k$ for some $k \in \mathbb{N}$. Therefore $9k^2 - 9kr + 3r^2 = 2007$ and we see that $3|r$. Let $r = 3s$. The equation becomes $k^2 - 3kr + 3r^2 = 223$. From our assumption $r \ge n/2$ we get $223 = k^2 - 3kr + 3r^2 = (k-r)(k-2r) + r^2 \le r^2$. Furthermore $4 \cdot 223 = (2k-3r)^2 + 3r^2 \ge 3r^2 \ge 3 \cdot 223$. Hence $r \in \{15, 16, 17\}$. For $r = 15$ and $r = 16$, $4 \cdot 223 - 3r^2$ is not a perfect square, and for $r = 17$ we get $(2k-3r)^2 = 25$ hence $2k - 3 \cdot 17 = \pm 5$. Both $k = 28$ and $k = 23$ lead to solutions $(n,r) = (84, 51)$ and $(n,r) = (69, 51)$.

11. Denote by $a_{k,1}, a_{k,2}, \dots, a_{k,n}$ the elements of A_k, and let $Q_k = \sum_{i=1}^{n} a_{k,i}^2$. Assume the contrary, that $|a_{k,i}| < n/2$ for all k, i. This means that the number of elements in $\bigcup_{k \in \mathbb{N}} A_k$ is finite. Hence there are different $p, q \in \mathbb{N}$ such that $A_p = A_q$. For any $I \subseteq \{1, 2, \dots, n\}$, denote $S_k(I) = \sum_{i \in I} a_{k,i}$. Let (I_k, J_k) be the partition that was chosen in constructing A_{k+1} from A_k.

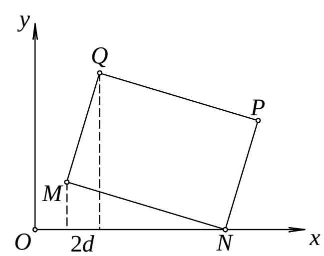
$$\begin{aligned} Q_{k+1}-Q_k &= \sum_{i\in I_k}\left((a_{k,i}+1)^2-(a_{k,i})^2\right)+\sum_{j\in J_k}\left((a_{k,j}-1)^2-(a_{k,j})^2\right) \\ &= n+2\left(S_k(I_k)-S_k(J_k)\right)=n-2\min_{I,J}|S_k(I)-S_k(J)|, \end{aligned}$$

where the last minimum is taken over all partitions (I,J) of the set $\{1,2,\dots,n\}$. However, for each k, it is easy to inductively build a partition (I',J') for which $|S_k(I')-S_k(J')|<n/2$. Take I_0 and J_0 to be empty sets, and assume we made the partition I_l, J_l of $\{1,\dots,l\}$ in such a way that $|S_k(I_l)-S_j(I_l)|<n/2$. Now, take

$$(I_{l+1},J_{l+1})=\begin{cases}(I_l\cup\{l+1\},J_l), & \text{if } S(I_l)\le S(J_l),\\ (I_l,J_l\cup\{l+1\}), & \text{if } S(I_l)>S(J_l).\end{cases}$$

Therefore $Q_{k+1}-Q_k>n-2\frac{n}{2}=0$ and Q_k is increasing. This contradicts the previously established fact that $A_p=A_q$ for some $p\neq q$.

12. Assume that $a>b$, $a=a_1d$, $b=b_1d$, $(a_1,b_1)=1$. There exist $p,q\in\mathbb{Z}$ such that $pa+qb=d$. We may assume that the pairs (S_n,S_{n+a}) and (S_n,S_{n+b}) are of different colors since otherwise the statement would follow immediately. By induction we get that $(S_n,S_{n+ua+vb})$ are of the same color if and only if $u+v$ is even. From $ab_1=ba_1$ we conclude $S_{ab_1}=S_{ba_1}$ which means that a_1 and b_1 must be of the same parity, hence they are both odd and $a_1\ge 3$. Furthermore, $pa_1+qb_1=1$ gives $2\nmid p+q$ which implies that the strips $S_{n+d}=S_{n+pa+qb}$ and S_n are of different colors. Now S_n and S_{n+2d} have the same color.



Consider the rectangle $MNPQ$ such that $MQ=NP=a$, $MN=PQ=b$ and the difference between the x coordinates of M and Q (and consequently N and P) is $2d$. It suffices to show that we can choose a rectangle in such a way that M and N are of the same color. Simple calculation shows that the difference of the x coordinates of M and N is $\tau=\frac{\sqrt{a_1^2-4}}{a_1}b\notin\mathbb{Q}$. Let s be one of longest single-colored (say red) segments on the x-axis. The translation s' of s to the left by τ has non-integer end-points, hence it can't be single-colored. Thus, there is a red point on s' – choose this point to be N. Other points are now easily determined.

Remark. We used the fact $a_1>2$ which is implied by $a\neq b$. The statement doesn't hold for squares (a counter-example is unit square).

13. Consider one of the cliques of maximal size $2n$ and put its members in a room X. Call these students *Π-students*. Put the others in the room Y. Let $d(X)$ and $d(Y)$ be the maximal sizes of cliques in X and Y in a given moment. If a student moves from X two Y, then $d(X)-d(Y)$ decreases by 1 or 2. Repeating this procedure we can make this difference 0 or -1.

Assume that it is -1 and $d(X)=l$, $d(Y)=l+1$. If the room Y contains a Π-student that doesn't belong to some of Y-cliques of the size $l+1$, after moving that student to X we will manage to have $d(X) = d(Y)$. Therefore assume that all $2n-l$ Π-students in Y belong to all Y-cliques of size $l+1$. Each such clique has to contain $2(l-n)+1 \geq 1$ *non-Π-students*. Take an arbitrary clique in Y of size $l+1$ and move a non-Π-student from it to X. Repeat this procedure as long as there are $l+1$ cliques in Y. We claim that $d(X)$ remains l after each such move. If not, consider $l+1$-clique in X. All its members would know all of $2n-l$ Π-students in Y. Together with them they would form $2n+1$ clique which is impossible. Hence we will end up with the configuration with $d(X) = d(Y) = l$.

14. Assume that $A_1, \dots, A_k$ are disjoint m-element subsets of $\{1, \dots, n\}$. Let

$$\mathscr{S} = \{S \subseteq \{1,\dots,n\} : S \cap A_i \neq \emptyset \text{ for all } i\} \quad \text{and}$$
$$\mathscr{T} = \{T \subseteq \{1,\dots,n\} : T \supseteq A_i \text{ for some } i, \text{ but } T \cap A_j = \emptyset \text{ for some } j\}.$$

For each $A \in \mathscr{S}$ and $B \in \mathscr{T}$ we have $A \cap B \neq \emptyset$. It suffices to prove:
Lemma. For $k = k(m) = \left[2^m \cdot \log \frac{3-\sqrt{5}}{2}\right]$ and $n = mk$ we have

$$\lim_{m\to\infty} \frac{|\mathscr{S}|}{2^n} = \lim_{m\to\infty} \frac{|\mathscr{T}|}{2^n} = \frac{3-\sqrt{5}}{2}.$$

Proof. For simplicity denote $\rho = \log \frac{3+\sqrt{5}}{2}$. For every i, each set in $\mathscr{S}$ must contain one of $2^m - 1$ non-empty subsets of A_i. Hence $|\mathscr{S}| = (2^m-1)^k$. Using the substitution $r = 2^m$ we get

$$\begin{aligned}\lim_{m\to\infty} \frac{|\mathscr{S}|}{2^{km}} &= \lim_{m\to\infty} \frac{(2^m-1)^k}{2^{mk}} = \lim_{r\to\infty} \frac{(r-1)^{\rho r}}{r^{\rho r}} \\ &= \lim_{r\to\infty} \left(1-\frac{1}{r}\right)^{\rho r} = e^{-\rho} = \frac{3-\sqrt{5}}{2}.\end{aligned}$$

In order to calculate $|\mathscr{T}|$, notice that $\mathscr{T} = \mathscr{U} \setminus (\mathscr{U} \cap \mathscr{S})$ where $\mathscr{U} = \{T \subseteq \{1,\dots,n\} : T \supseteq A_i \text{ for some } i\}$. Obviously, $|\mathscr{U}^c| = (2^m-1)^k$ which gives us $|\mathscr{U}| = 2^{mk} - (2^m-1)^k$. Furthermore, $|\mathscr{U} \cap \mathscr{S}| = |\mathscr{S}| - |\mathscr{S} \setminus \mathscr{U}|$. From $\mathscr{S} \setminus \mathscr{U} = \{T \subseteq \{1,\dots,n\} : T \cap A_i \neq 0 \text{ and } T \cap A_i \neq A_i \text{ for all } i\}$ we get $|\mathscr{S} \setminus \mathscr{U}| = (2^m-2)^k$. Using the substitution $r = 2^m$ we get

$$\begin{aligned}\lim_{m\to\infty} \frac{|\mathscr{T}|}{2^n} &= \lim_{m\to\infty} \frac{2^{km} - 2(2^m-1)^k + (2^m-2)^k}{2^{mk}} \\ &= 1 - 2e^{-\rho} + \lim_{r\to\infty} \frac{(r-2)^{\rho r}}{r^{\rho r}} = 1 - 2e^{-\rho} + \lim_{r\to\infty}\left(1-\frac{2}{r}\right)^{\rho r} \\ &= 1 - 2e^{-\rho} + e^{-2\rho} = e^{-\rho} = \frac{3-\sqrt{5}}{2}.\end{aligned}$$

15. For each vertex V of P consider all good triangles whose one vertex is V. All the vertices of these triangles belong to the unit circle centered at V. Label them counter-clockwise as $V_1, \dots, V_i$. Denote by $f(V)$ and $l(V)$ the first and the last of

these vertices ($f(V) = V_1$, $l(V) = V_i$). Denote by $T_f(V)$ the good triangle with vertices V and $f(V)$. $T_l(V)$ is defined analogously. We call $T_f(V)$ and $T_l(V)$ the triangles *associated* with V (they might be the same). Let α be the total number of associated triangles, and t the total number of good triangles. It is enough to prove that each good triangle is associated with at least three vertices. Indeed this would imply $3t \le \alpha \le 2n$. It suffices to show that for an arbitrary good triangle ABC oriented counter-clockwise we have $A = l(B)$ or $A = f(C)$. Assume that $A \ne l(B)$ and $A \ne f(C)$. Then $l(B)$ and $f(C)$ would belong to the half plane $[BC,A$. Define $A' = l(B)$ if $\angle ABl(B) \le 60^\circ$. If this angle is bigger than 60° define A' to be the third vertex of $T_l(B)$. Similarly we define A''. We have $\angle ABA' < 60^\circ$, $\angle ACA'' < 60^\circ$, hence A belongs to the interior of the rectangle $A'BCA''$ and P can't be convex. This concludes the proof of our claim.

Remark. It is easy to refine the proof to show that $t \le [\frac{2}{3}(n-1)]$. This result is sharp and the example of $3k+1$-gon with $2k$ good triangles is not hard to construct: rotate a rhombus $ABCD$ ($AB = BC = DA = 1$) around A by small angles k times.

16. Let O be the circumcenter of the triangle ABC. We know that $\angle CPK = \angle CQL$ hence $\frac{S_{RPK}}{S_{RQL}} = \frac{RP \cdot PK}{RQ \cdot QL}$. Since $\triangle PKC \sim \triangle QLC$ we have $\frac{PK}{QL} = \frac{PC}{QC}$. Since ROC is isosceles and $\angle OPR = \angle OQC$ we get $\triangle ROQ \cong \triangle COP$ and $RQ = PC$. This finally implies $\frac{S_{RPK}}{S_{RQL}} = 1$.

17. Let Y be the midpoint of BT. Then $MY \| CT$ and $TY \perp XY$ hence T, Y, M, X belong to a circle. Thus $\angle MTB - \angle CTM = \angle MXY - \angle YMT = \angle MXY - \angle TXY = \angle MXY - \angle YXB = \angle MXB = \angle BAM$.

18. Let X be the point on the line PQ such that $XC \| AQ$. Then $XC : AQ = CP : PA = BC : AD$ which implies $\triangle BCX \sim \triangle DAQ$. Hence $\angle DAQ = \angle BCX$ and $\angle BXC = \angle DQA = \angle BQC$. Therefore B, C, Q, X belong to a circle which implies that $\angle BCX = \angle BQX = \angle BQP$.

19. Let K and L be the midpoints of FC and CG respectively. Then $EK \perp CD$ and $EL \perp BC$ hence KL is the Simson's line of the triangle BCD and intersects BD at point M such that $EM \perp BD$. We also have that $KL \| l$ and KL bisects the side CA of $\triangle ACG$. Hence KL passes through the intersection of the diagonals of $ABCD$ thus KL bisects BD. Therefore $DM = MB$ and DEB is isosceles. From $\triangle DEB \sim \triangle KEL$ we get that $EK = EL$ hence $CF = CG$. Thus $\angle DAF = \angle FGC = \angle GFC = \angle FAB$.

20. Denote by A_0, B_0, and C_0 the given intersection points. Applying the Pascal's theorem to the points $APCC'BA'$ gives us that $B_0 \in C_1A_1$. Similarly we get $C_0 \in A_1B_1$ and $A_0 \in B_1C_1$. From $B_0A_1 \| AB_1$ we get $\frac{C_0B_0}{C_0A} = \frac{C_0A_1}{C_0B_1}$. Since $BA_1 \| B1A_0$ we get $\frac{C_0A_1}{C_0B_1} = \frac{C_0B}{C_0A_0}$. Hence $\frac{C_0B_0}{C_0A} = \frac{C_0B}{C_0A_0}$ or $C_0B_0 \cdot C_0A_0 = C_0A \cdot C_0B$. Therefore $S_{A_0B_0C_0} = S_{ABC_0}$. However, $S_{ABC_0} = S_{ABB_1}$ (because $A_1B_1 \| AB$) hence $S_{A_0B_0C_0} = S_{ABB_1} = \frac{1}{2} S_{ABC}$.

21. Let us prove that $S_1 \ge S$, i.e. that $k \le 1$.

Lemma. If X', Y', Z' are the points on the sides YZ, ZX, XY of $\triangle XYZ$ then $S_{X'Y'Z'} \geq \min\{S_{XY'Z'}, S_{YZ'X'}, S_{ZX'Y'}\}$.

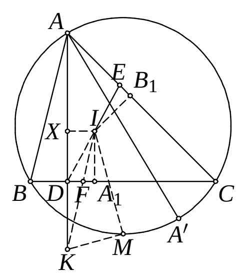
Proof. Denote by X_1, Y_1, Z_1 the midpoints of YZ, ZX, XY. If two of X', Y', Z' belong to one of the triangles XY_1Z_1, YZ_1X_1, ZX_1Y_1, then the statement follows immediately. Indeed, if Y', $Z' \in XY_1Z_1$, then $Y'Z'$ intersects the altitude from X to YZ at the point Q inside $\triangle XY_1Z_1$, which forces $d(X, Z'Y') \leq d(X, Z_1Y_1) = \frac{1}{2}d(X, YZ) \leq d(X', Y'Z')$. Assume now, w.l.o.g. that $X' \in X_1Z$, $Y' \in Y_1X$, $Z' \in Z_1Y$. Then $d(Z', X'Y') > d(Z_1, X'Y')$, hence $S_{X'Y'Z'} > S_{X'Y'Z_1}$. Similarly $S_{X'Y'Z_1} > S_{X'Y_1Z_1}$. Since $S_{X'Y_1Z_1} = S_{X_1Y_1Z_1}$ we have $S_{X'Y'Z'} > S_{X_1Y_1Z_1} = \frac{1}{4}S_{XYZ}$, hence $S_{X'Y'Z'} > \min\{S_{XY'Z'}, S_{YZ'X'}, S_{ZX'Y'}\}$.

If $S_{A_1B_1C_1} \geq \min\{S_{A_1BB_1}, S_{B_1CC_1}\}$ and $S_{A_1C_1D_1} \geq \min\{S_{C_1DD_1}, S_{D_1AA_1}\}$ then the problem is trivial. The same holds for $S_{A_1B_1D_1}$ and $S_{B_1C_1D_1}$.

Without loss of generality assume that $S_{A_1B_1C_1} < \min\{S_{A_1BB_1}, S_{B_1CC_1}\}$ and $S_{A_1B_1D_1} < \min\{S_{A_1BB_1}, S_{AA_1D_1}\}$. Assume also that $S_{A_1B_1C_1} \leq S_{A_1B_1D_1}$. Then the line C_1D_1 intersects the ray $(BC$ at some point U. The lines AB and CD can't intersect at a point that is on the same side of A_1C_1 as B_1. Otherwise, any line through C_1 that intersects $(BA$ and $(BC$, say at M and N, would force $S_{C_1B_1N} > S_{A_1B_1C_1}$ and $S_{A_1BB_1} > S_{A_1B_1C_1}$. The lemma would imply that $S_{MA_1C_1} \leq S_{A_1B_1C_1}$. This is impossible since we can make $S_{MA_1C_1}$ arbitrarily large. Therefore $C_1D_1 \cap (BA = V$. Applying the lemma to $\triangle VBU$ we get $S_{A_1B_1C_1} \geq S_{VA_1C_1} > S_{A_1C_1D_1}$, a contradiction.

To show that the constant $k = 1$ is the best possible, consider the cases close to the degenerate one in which ACD is a triangle, D_1, C_1 the midpoints of AD and CD, and $B = A_1 = B_1$ the midpoint of AC.

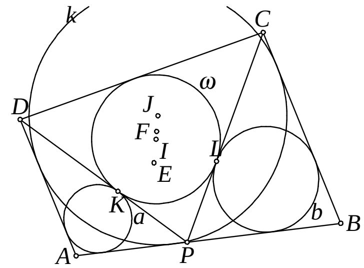
22. We will prove that $\angle KID = \frac{\beta-\gamma}{2}$. Let AA' be the diameter of the circumcircle k of $\triangle ABC$. Denote by M the intersection of AI with k.



Since $A'M \perp AM$ we have that K, M, and A' are colinear. Let A_1, B_1, and X be the feet of perpendiculars from I to BC, CA, and AD. Then $\triangle XIB_1 \sim \triangle A'MB$ since the corresponding angles are equal (this is easy to verify). Since $MB = MI = MC$ we conclude $IM : KM = BM : MA' = IB_1 : IX = XD : IX$ hence $\triangle KIM \sim \triangle IDX$. Therefore $\angle KID = \angle XIM - (\angle XID + \angle KIM) = \angle XIM - 90^\circ = 180^\circ - \angle AA'M - 90^\circ = \angle MAA' = \frac{\beta-\gamma}{2}$.

Assume now that $IE = IF$. Since $\beta > \gamma$ we have that A_1 belongs to the segment FC and hence $\angle C = \angle DIA_1 + \angle EIB_1 = \angle DIF + 2\angle FIA_1$. This is equivalent to $2\angle FIA_1 = \gamma - \frac{\beta-\gamma}{2}$ thus $\beta < 3\gamma$.

23. Let J be the center of circle k tangent to the lines AB, DA, and BC. Denote by a and b the incircles of $\triangle ADP$ and $\triangle BCP$.



First we will prove that $F \in IJ$. A is the center of homothethy that maps a to k; K is the center of negative homothethy that maps a to ω; and denote by $\hat{F}$ the center of negative homothethy that maps ω to k. By the consequence of the Desargue's theorem we have that A, K, and $\hat{F}$ are colinear. Similarly we prove that $\hat{F} \in BL$ therefore $F = \hat{F}$ and $F \in IJ$. Now we will prove that $E \in IJ$. Denote by X and Y the centers of inversions that map a and b to ω. Comparing the lengths of the tangents from A, B, C, D to the circles k and a we get that $AP + DC = AD + PC$. Hence there exists a circle d inscribed in $APCD$. Let X be the center of homothethy that maps a to ω. Using the same consequence of the Desargue's theorem we see that A, C, and X are colinear. Consider the circles a, ω and k again. A is the center of homothethy that maps a to k and X is the center of homothethy that maps a to ω. Therefore XA contains the center $\hat{E}$ of homothethy with positive coefficient that maps ω to k. Similarly $\hat{E} \in BX$, hence $\hat{E} = E$ and $E \in IJ$.

24. $k^4 + n^2$ must be even because $7^k - 3^n$ is even. If both k and n are odd, then $k^4 + n^2 \equiv 2 \pmod 4$ while $7^k - 3^n \equiv 7 - 3 \equiv 0 \pmod 4$. Assume that $k = 2k'$ and $n = 2n'$. Then $7^k - 3^n = \frac{7^{k'} - 3^{n'}}{2} \cdot 2(7^{k'} + 3^{n'})$ and $2(7^{k'} + 3^{n'})$ must be a divisor of $2(8k'^4 + 2n'^2)$ hence $7^{k'} + 3^{n'} \le 8k'^2 + 2n'^2$. It is easy to prove by induction that $7^{k'} > 8k'^4$ for $k' \ge 4$ and $3^{n'} > 2n'^2$ for $n' \ge 1$. Therefore $k' \in \{1, 2, 3\}$.
For $k' = 1$ we must have $7 + 3^{n'} | 8 + 2n'^2$. An easy induction gives $7 + 3^{n'} > 8 + 2n'^2$ for $n' \ge 3$. Hence $n' \le 2$. For $n' = 1$ we get $(k, n) = (2, 2)$ which doesn't satisfy the given conditions. $n' = 2$ implies $(k, n) = (2, 4)$ which is a solution.
Assume now that $k' = 2$. Then $|7^k - 3^n| = |7^2 - 3^{n'}| \cdot (7^2 + 3^{n'}) \ge 22 \cdot (49 + 3^{n'}) > 4^4 + 4n'^2$. This contradiction proves that $k' \ne 2$.
If we assume that $k' = 3$, then $|7^k - 3^n| = |7^3 - 3^{n'}| \cdot (7^3 + 3^{n'}) = |343 - 3^{n'}| \cdot (7^3 + 3^{n'}) \ge 100 \cdot (7^3 + 3^{n'}) > 6^4 + 4n'^2$. This is again a contradiction.
Thus $(k, n) = (2, 4)$ is the only solution.

25. Assume that $b = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ where $p_1, \dots, p_l$ are prime numbers. Since $b - a_{b^2}^n$ is divisible by b^2 we get that $p_i^{\alpha_i} | a_{b^2}^n$ but $p_i^{\alpha_i + 1} \nmid a_{b^2}^n$ for each i. This implies that $n | \alpha_i$ for each i hence b is a complete nth power.

26. Set Z of integers will be called *good* if $47 \nmid a - b + c - d + e$ for any $a, b, c, d, e \in Z$. Notice that the set $G = \{-9, -7, -5, -3, -1, 1, 3, 5, 7, 9\}$ is good. For each integer $k \in \{1, 2, \dots, 46\}$ the set $G_k = \{x \in X | \exists g \in G : kx \equiv g \pmod{47}\}$ is good as well. Indeed, if $a_i \in G_k$ $(1 \le i \le 5)$ are such that $47 | a_1 - a_2 + a_3 - a_4 + a_5$ then $47 | ka_1 - ka_2 + ka_3 - ka_4 + ka_5$. There are elements $b_i \in G$ for which $ka_i \equiv b_i \pmod{47}$ which is impossible. Each element of x is contained in exactly 10 of the sets G_k hence $10|X| = \sum_{i=1}^{46} |A_k|$ therefore $|A_k| > 2173 > 2007$ for at least one k.

27. The difference of the two binomial coefficients can be written as

$$D=\binom{2^{k+1}}{2^k}-\binom{2^k}{2^{k-1}} = \frac{(2^{k+1})!}{(2^k)!\cdot(2^k)!}-\frac{1}{(2^k)!}\cdot\left(\frac{(2^k)!}{(2^{k-1})!}\right)^2$$
$$=\frac{2^{2^k}}{(2^k)!}\cdot(2^{k+1}-1)!!-\frac{2^{2^{k-1}\cdot 2}}{(2^k)!}\cdot((2^k-1)!!)^2$$
$$=\frac{2^{2^k}\cdot(2^k-1)!!}{(2^k)!}\cdot P(2^k),$$

for $P(x)=(x+1)(x+3)\cdots(x+2^k-1)-(x-1)\cdot(x-3)\cdots(x-2^k+1)$. The exponenet of 2 in $(2^k)!$ is equal to 2^k-1 hence the exponent of 2 in D is by 1 bigger then the exponent of 2 in $P(2^k)$. Since $P(-x)=-P(x)$ we get $P(x)=\sum_{i=1}^{2^{k-1}}c_ix^{2i-1}$. Being the coefficient near x, c_1 satisfies:

$$c_1=2\cdot(2^k-1)!!\cdot\sum_{i=1}^{2^{k-1}}\frac{1}{2i-1}=(2^k-1)!!\cdot\sum_{i=1}^{2^{k-1}}\left(\frac{1}{2i-1}+\frac{1}{2^k-2i+1}\right)$$
$$=2^k\cdot\sum_{i=1}^{2^{k-1}}\frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)}.$$

Let a_i be the solution of $a_i\cdot(2i-1)\equiv 1 \pmod{2^k}$. Then

$$\sum_{i=1}^{2^{k-1}}\frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)}\equiv-\sum_{i=1}^{2^{k-1}}(2^k-1)!!\cdot a_i^2$$
$$=-(2^k-1)!!\sum_{i=1}^{2^{k-1}}(2i-1)^2$$
$$=-(2^k-1)!!\frac{2^{k-1}(2^k+1)(2^k-1)}{3}$$
$$\equiv 2^{k-1} \pmod{2^k}.$$

The exponent of 2 in c_1 has to be $2k-1$. Now $P(2^k)=c_1\cdot 2^k+2^{3k}Q(2^k)$ for some polynomial Q. Clearly, $2^{3k-1}|P(2^k)$ but $2^{3k}\nmid P(2^k)$. Finally, the exponent of 2 in D is equal to $3k$.

28. Fix a prime number p. Let $d\in\mathbb{N}$ be the smallest number for which $p|f(d)$ (it exists because f is surjective). By induction we have $p|f(kd)$ for every $k\in\mathbb{N}$. If $p|f(x)$ but $d\nmid x$, from the minimality of d we conclude $x=kd+r$, where $r\in\{1,2,\dots,d-1\}$. Now we have $p|f(kd)+f(r)$ hence $p|f(r)$ which is impossible. Therefore $d|x\Leftrightarrow p|f(x)$.
If $x\equiv y \pmod d$, let $D>x$ be some number such that $d|D$. Then $d|(y-x+D)$ hence $p|f(y-x+D)$, and $p|f(y)+f(D-x)$. Since $p|f(x+D-x)$ we get $p|f(x)+f(D-x)$. We obtained $x\equiv y \pmod d \Rightarrow f(x)\equiv f(y) \pmod p$.
Now assume that $f(x)\equiv f(y) \pmod p$. Taking the same D as above and assuming that $y>x$ we get $0\equiv f(x)+f(D-x)\equiv f(y)+f(D-x)\equiv f(y+$

$D-x) \equiv f(y-x)+f(D) \equiv f(y-x) \pmod p$. This implies that $d|y-x$, thus $x \equiv y \pmod d \Leftrightarrow f(x) \equiv f(y) \pmod p$.
Now we know that $f(1),\dots,f(d)$ have different residues modulo p hence $p \geq d$. Since f is surjective there are numbers $x_1,\dots,x_p$ such that $f(x_1)=1,\cdots,f(x_p)=p$. They all give different residues modulo p hence $x_1,\dots,x_p$ must give p distinct residues modulo d implying $p=d$.
Now we have $p|x \Leftrightarrow p|f(x)$ and $x \equiv y \pmod p \Leftrightarrow f(x) \equiv f(y) \pmod p$ for every $x,y \in \mathbb{N}$ and every prime number p. Since no prime divides 1 we must have $f(1)=1$. We will prove by induction that $f(n)=n$. Assume that $f(k)=k$ for every $k<n$. If $f(n)>n$ then $f(n)-n+1 \geq 2$ will have a prime factor p. This is impossible because $f(n) \equiv n-1 = f(n-1) \pmod p$, hence $n \equiv n-1 \pmod p$. If $f(n)<n$, let p be a prime factor of $n-f(n)+1 \geq 2$. Now we have $n \equiv f(n)-1 \pmod p$ and $f(n) \equiv f(f(n)-1) = f(n)-1 \pmod p$, contradiction. Thus $f(n)=n$ is the only possible solution. It is easy to verify that this f satisfies the given conditions.

29. The statement will follow from the following lemma applied to $x=k$ and $y=2n$.
Lemma. Given two positive integers x and y, the number $4xy-1$ divides the number $(4x^2-1)^2$ if and only if $x=y$.
Proof. If $x=y$ it is obvious that $4xy-1|(4x^2-1)^2$. Assume that there is a pair (x,y) of two distinct positive integers such that $4xy-1|(4x^2-1)$. Choose such a pair for which $2x+y$ is minimal. From $(4y^2-1)^2 \equiv (4y^2-(4xy)^2)^2 \equiv 16y^2(4x^2-1)^2 \equiv 0 \pmod{4xy-1}$ we get that (y,x) is such pair as well hence $2y+x>2x+y$ and $y>x$.
Assume that $(4x^2-1)^2 = k\cdot(4xy-1)$. Multiplying $4xy-1 \equiv -1 \pmod{4x}$ by k we get $(4x^2-1)^2 \equiv -k \pmod{4x}$ hence $k=4xl-1$ for some positive integer l. However, this means that $4xl-1|(4x^2-1)^2$ and since $y>x$ we must have $l<x$ implying $2l+y<2x+y$ and this contradicts the minimality of (x,y).
Remark: Using the same method one can prove the more general theorem: If $k>1$ is an integer, then $kab-1|(ka^2-1)^2 \Leftrightarrow a=b$.

30. Denote by $f_i(n)$ the remainder when $v_{p_i}(n)$ is divided by d. Let $f(n)=(f_1(n),\dots,f_k(n))$. Consider the sequence of integers n_j defined inductively as $n_1=1$ and $n_{j+1}=(p_1\cdots p_k)^{n_j}$. Let us first prove that $v_p(r+lp^m)=v_p(r)+v_p(lp^m)$ for $r<p^m$. This follows from $(r+lp^m)! = (lp^m)!\cdot(lp^m+1)\cdots(lp^m+r)$ and for each $i<p^m$ the exponent of p in lp^m+i is equal to the exponent of p in i.
If $j_1<j_2<\cdots<j_u$, the exponent of p_i in each of $n_{j_2},\dots,n_{j_u}$ is bigger than n_{j_1} hence $f_i(n_{j_1}+n_{j_2}+\cdots+n_{j_u}) \equiv f_i(n_{j_1})+f_i(n_{j_2}+\cdots+n_{j_u}) \pmod d$. Continuing by induction we get $f_i(n_{j_1}+\cdots+n_{j_u}) \equiv f_i(n_{j_1})+\cdots+f_i(n_{j_u}) \pmod d$.
Since the range of f has at most $(d+1)^k$ elements we see that there is an infinite subsequence of n_i on which f is constant. Then for any d elements $n_{l_1},\dots,n_{l_d}$ of this subsequence we have $f(n_{l_1}+\cdots+n_{l_d}) \equiv f(n_{l_1})+\cdots+f(n_{l_d}) \equiv df(n_{l_1}) \equiv (0,0,\dots,0) \pmod d$.

A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathscr{B}(A,B,C)$, $A-B-C$: indicates the relation of *betweenness*, i.e., that B is between A and C (this automatically means that A,B,C are different collinear points).
- $A = l_1 \cap l_2$: indicates that A is the intersection point of the lines l_1 and l_2.
- AB: line through A and B, segment AB, length of segment AB (depending on context).
- $[AB$: ray starting in A and containing B.
- $(AB$: ray starting in A and containing B, but without the point A.
- (AB): open interval AB, set of points between A and B.
- $[AB]$: closed interval AB, segment AB, $(AB) \cup \{A,B\}$.
- $(AB]$: semiopen interval AB, closed at B and open at A, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- ABC: plane determined by points A,B,C, triangle ABC ($\triangle ABC$) (depending on context).
- $[AB,C$: half-plane consisting of line AB and all points in the plane on the same side of AB as C.
- $(AB,C$: $[AB,C$ without the line AB.

- $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$, $\overrightarrow{a} \cdot \overrightarrow{b}$: scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$.
- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle ABC (unless otherwise indicated).
- $k(O, r)$: circle k with center O and radius r.
- $d(A, p)$: distance from point A to line p.
- $S_{A_1A_2\ldots A_n}$, $[A_1A_2\ldots A_n]$: area of n-gon $A_1A_2\ldots A_n$ (special case for $n = 3$, S_{ABC}: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo n, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo p, p being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in x with integer and real coefficients respectively.
- R^*: the set of nonzero elements of a ring R.
- $R[\alpha]$, $R(\alpha)$, where α is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- X_0: $X \cup \{0\}$ for X such that $0 \notin X$.
- X^+, X^-, $aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to x.
- $\lceil x \rceil$: the smallest integer greater than or equal to x.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points AB, the absolute value of the number x, the number of elements of the set S (respectively).
- (x, y), (m, n), (a, b): (ordered) pair x and y, the greatest common divisor of integers m and n, the open interval between real numbers a and b (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).

- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

B

Codes of the Countries of Origin

ARG	Argentina
ARM	Armenia
AUS	Australia
AUT	Austria
BEL	Belgium
BLR	Belarus
BRA	Brazil
BUL	Bulgaria
CAN	Canada
CHN	China
COL	Colombia
CRO	Croatia
CUB	Cuba
CYP	Cyprus
CZE	Czech Republic
CZS	Czechoslovakia
EST	Estonia
FIN	Finland
FRA	France
FRG	Germany, FR
GBR	United Kingdom
GDR	Germany, DR
GEO	Georgia
GER	Germany
GRE	Greece
HKG	Hong Kong
HUN	Hungary
ICE	Iceland
INA	Indonesia
IND	India
IRE	Ireland
IRN	Iran
ISR	Israel
ITA	Italy
JAP	Japan
KAZ	Kazakhstan
KOR	Korea, South
KUW	Kuwait
LAT	Latvia
LIT	Lithuania
LUX	Luxembourg
MCD	Macedonia
MEX	Mexico
MON	Mongolia
MOR	Morocco
NET	Netherlands
NOR	Norway
NZL	New Zealand
PER	Peru
PHI	Philippines
POL	Poland
POR	Portugal
PRK	Korea, North
PUR	Puerto Rico
ROM	Romania
RUS	Russia
SAF	South Africa
SER	Serbia
SIN	Singapore
SLO	Slovenia
SMN	Serbia and Montenegro
SPA	Spain
SVK	Slovakia
SWE	Sweden
THA	Thailand
TUN	Tunisia
TUR	Turkey
TWN	Taiwan
UKR	Ukraine
USA	United States
USS	Soviet Union
UZB	Uzbekistan
VIE	Vietnam
YUG	Yugoslavia