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IMO Shortlist 2008

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Problems

1.1 The Forty-Nineth IMO
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1.1.1 Contest Problems

First Day (July 16)

1. An acute-angled triangle ABC has orthocenter H. The circle passing through H with center the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with center the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with center the midpoint of AB intersects the line AB at C_1 and C_2. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

2. (a) Prove that
$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$
for all real numbers x, y, z, each different from 1, and satisfying $xyz=1$.
(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, satisfying $xyz=1$.

3. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor which is greater than $2n+\sqrt{2n}$.

Second Day (July 17)

4. Find all functions $f:(0,+\infty)\to(0,+\infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that
$$\frac{(f(w))^2+(f(x))^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$
for all positive real numbers w, x, y, z, satisfying $wx=yz$.

5. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequence of *steps*: at each step one of the lamps is switched (from on to off or from off to on).
Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.
Let M be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.
Determine the ratio N/M.

6. Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

1.1.2 Shortlisted Problems

1. **A1 (KOR)** IMO4 Find all functions $f : (0, +\infty) \to (0, +\infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that
$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$
for all positive real numbers w, x, y, z, satisfying $wx = yz$.

2. **A2 (AUT)** IMO2
(a) Prove that
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$
for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.
(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, satisfying $xyz = 1$.

3. **A3 (NET)** Let $S \subseteq \mathbb{R}$ be a set of real number. We say that a pair (f, g) of functions from S to S is a *Spanish Couple* on S, if they satisfy the following conditions:
(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all $x, y \in S$ with $x < y$;
(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.
Decide whether there exists a Spanish Couple
(a) on the set $S = \mathbb{N}$ of positive integers;
(b) on the set $S = \{a - 1/b : a, b \in \mathbb{N}\}$.

4. **A4 (AUT)** For an integer m, denote by $t(m)$ the unique number in $\{1,2,3\}$ such that $m+t(m)$ is a multiple of 3. A function $f:\mathbb{Z}\to\mathbb{Z}$ satisfies $f(-1)=0$, $f(0)=1$, $f(1)=-1$ and
$$f(2^n+m)=f(2^n-t(m))-f(m) \quad \text{for all integers } m,n\geq 0 \text{ with } 2^n>m.$$
Prove that $f(3p)\geq 0$ holds for all integers $p\geq 0$.

5. **A5 (SVK)** Let a, b, c, d be positive real numbers such that
$$abcd=1 \quad \text{and} \quad a+b+c+d>\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}.$$
Prove that
$$a+b+c+d<\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

6. **A6 (LIT)** Let $f:\mathbb{R}\to\mathbb{N}$ be a functions which satisfies
$$f\left(x+\frac{1}{f(y)}\right)=f\left(y+\frac{1}{f(x)}\right), \qquad \text{for all } x,y\in\mathbb{R}.$$
Prove that there is a positive integer which is not a value of f.

7. **A7 (GER)** Prove that for any four positive real numbers a, b, c, d the inequality
$$\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b}\geq 0$$
holds. Determine all cases of equality.

8. **C1 (NET)** A *box* is a rectangle in the plane whose sides are parallel to the coordinate axes and have positive lengths. Two boxes *intersect* if they have a common point in their interior or on their boundary.
Find the largest n for which there exist n boxes $B_1, \dots, B_n$ such that B_i and B_j intersect if and only if $i\not\equiv j\pm 1 \pmod{n}$.

9. **C2 (SER)** For every positive integer n determine the number of permutations $(a_1,\dots,a_n)$ of the set $\{1,2,\dots,n\}$ with the following property:
$$2(a_1+a_2+\cdots+a_k) \text{ is divisible by } k \text{ for } k=1,2,\dots,n.$$

10. **C3 (PER)** Consider the set S of all points with integer coordinates in the coordinate plane. For a positive integer k, two distinct points A, $B\in S$ will be called *k-friends* if there is a point $C\in S$ such that the area of the triangle ABC is equal to k. A set $T\subseteq S$ will be called a *k-clique* if every two points in T are k-friends. Find the least positive integer k for which there exists a k-clique with more than 200 elements.

11. **C4 (FRA)** IMO5 Let n and k be positive integers with $k\geq n$ and $k-n$ an even number. Let $2n$ lamps labelled 1, 2, …, $2n$ be given, each of which can be either

on or *off*. Initially all the lamps are off. We consider sequence of *steps*: at each step one of the lamps is switched (from on to off or from off to on).
Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.
Let M be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.
Determine the ratio N/M.

12. **C5 (RUS)** Let $S=\{x_1,x_2,\dots,x_{k+l}\}$ be a $k+l$-element set of real numbers contained in the interval $[0,1]$; k and l are positive integers. A k-element subset $A\subseteq S$ is called *nice* if
$$\left|\frac{1}{k}\sum_{x_i\in A}x_i-\frac{1}{l}\sum_{x_j\in S\setminus A}x_j\right|\le\frac{k+l}{2kl}.$$
Prove that the number of nice subsets is at least $\frac{2}{k+l}\cdot\binom{k+l}{k}$.

13. **C6 (NET)** For $n\ge 2$, let $S_1, S_2, \dots, S_{2^n}$ be 2^n subsets of $A=\{1,2,3,\dots,2^{n+1}\}$ that satisfy the following property: There do not exist indeces a and b with $a<b$ and elements $x,y,z\in A$ with $x<y<z$ such that $y,z\in S_a$ and $x,z\in S_b$. Prove that at least one of the sets $S_1, S_2, \dots, S_{2^n}$ contains no more than $4n$ elements.

14. **G1 (RUS)** IMO1 An acute-angled triangle ABC has orthocenter H. The circle passing through H with center the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with center the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with center the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

15. **G2 (LUX)** Given a trapeziod $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside the segment BC, and F inside the segment AD, such that $\angle DAE=\angle CBF$. Denote by I the intersection point of CD and EF, and by J the intersection point of AB and EF. Let K be the midpoint of the segment EF. Assume K does not line on the lines AB and CD.
Prove that I belongs to the circumcircle of $\triangle ABK$ if and only if K belongs to the circumcircle of $\triangle CDJ$.

16. **G3 (PER)** Let $ABCD$ be a convex quadrilateral and let P and Q be the points such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE=\angle QDE$ and $\angle PBE=\angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

17. **G4 (IRN)** Let BE and CF be the altitudes in an acute triangle ABC. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that the lines PE and QF intersect on the circumcircle of $\triangle AEF$.

18. **G5 (NET)** Let k and n be integers with $0 \le k \le n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersection points of lines in L. Let O be a point in the plane not lying on any line of L.
A point $X \in I$ is colored in red if the open line segment (OX) intersects at most k lines from L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

19. **G6 (SER)** Let $ABCD$ be a convex quadrilateral. Prove that there exists a point P inside the quadrilateral such that
$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$
if and only if the diagonals AC and BD are perpendicular.

20. **G7 (RUS)** IMO6 Let $ABCD$ be a convex quadrilateral with $|BA| \ne |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

21. **N1 (AUS)** Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations
$$a^n + pb = b^n + pc = c^n + pa,$$
then $a = b = c$.

22. **N2 (IRN)** Let $a_1, a_2, \dots, a_n$ be distinct positive integers, $n \ge 3$. Prove that there exist distinct indeces i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \dots, 3a_n$.

23. **N3 (IRN)** Let a_0, a_1, a_2 be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term, i.e. $(a_i, a_{i+1}) > a_{i-1}$ for all $i \ge 1$. Prove that $a_n \ge 2^n$ for all $n \ge 0$.

24. **N4 (SER)** Let n be a positive integer. Show that the numbers
$$\binom{2^n-1}{0}, \binom{2^n-1}{1}, \binom{2^n-1}{2}, \dots, \binom{2^n-1}{2^{n-1}-1}$$
are congruent modulo 2^n to $1, 3, 5, \dots, 2^n-1$ in some order.

25. **N5 (FRA)** For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties:
(i) $d(f(x)) = x$ for all $x \in \mathbb{N}$;
(ii) $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.

26. **N6 (LIT)** IMO3 Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor which is greater than $2n + \sqrt{2n}$.

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Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2008

1. For $x=y=z=w$ the functional equation gives $f(x)^2=f(x^2)$ for all $x\in\mathbb{R}_+$. In particular, $f(1)=1$. Setting $\sqrt{w}$, $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ in the equation yields

$$\frac{f(w)+f(x)}{f(y)+f(z)}=\frac{w+x}{y+z},\ \text{ whenever } wx=yz.$$

Choosing $z=1$ we get $w=y/x$ and $f(y/x)+f(x)=\left(\frac{y}{x}+x\right)\cdot\frac{f(y)+1}{y+1}$. Now if we place $y=x^2$ we get $f(x)=x\cdot\frac{f(x)^2+1}{x^2+1}$ which is equivalent to $(f(x)-x)(f(x)-\frac{1}{x})=0$. Assume that there are $x,w\in\mathbb{R}_+\setminus\{1\}$ such that $f(x)=x$ and $f(w)=\frac{1}{w}$. Choosing $y=z=\sqrt{wx}$ and placing in the equation implies $\frac{1}{w}+x=(w+x)\cdot\frac{f(\sqrt{wx})}{\sqrt{wx}}$. If $f(\sqrt{wx})=\sqrt{wx}$ then we have $w=1$. Otherwise, if $f(\sqrt{wx})=1/\sqrt{wx}$ we get $x=1$, contrary to our assumption. Therefore we either have $f(x)=x$ for all $x\in\mathbb{R}_+$ or $f(x)=\frac{1}{x}$ for all $x\in\mathbb{R}_+$. It is easy to verify that both functions satisfy the original equation.

2. (a) Substituting $a=\frac{x}{x-1}$, $b=\frac{y}{y-1}$, $c=\frac{z}{z-1}$ the inequality becomes equivalent to $a^2+b^2+c^2\geq 1$ while the constraint becomes $a+b+c=ab+bc+ca+1$. The last equation is equivalent to $2(a+b+c)=(a+b+c)^2-(a^2+b^2+c^2)+2=(a+b+c)^2+1-[(a^2+b^2+c^2)-1]$, or $[(a+b+c)-1]^2=(a^2+b^2+c^2)-1$ which immediately implies that $a^2+b^2+c^2\geq 1$.

   (b) The equality holds if and only if $a+b+c=1$ and $a+b+c=ab+bc+ca+1$. Expressing $c=1-a-b$ yields $-ab+a+b-a^2-b^2=0$. It suffices to prove that there are infinitely many rational numbers a for which the quadratic equation $b^2-(a-1)b-a(a-1)=0$ has a rational solution b. The last quadratic equation has a rational solution if and only if its discriminant $(a-1)^2-4a(a-1)=(1-a)(1+3a)$ is a square of a rational number. We want to find infinitely many rational numbers $\frac{p}{q}$ such that $\left(1-\frac{p}{q}\right)\left(1+3\frac{p}{q}\right)$ is a square of a rational number, which is equivalent to $(q-p)(q+3p)$ being a square of an integer. However, for each $m,n\in\mathbb{N}$ the system $q-p=(2m+1)^2$ and $q+3p=(2n+1)^2$ has a solution $p=n^2+n-m^2-m$, $q=n^2+n+3m^2+3m+1$. Keeping m fixed, and increasing n would guarantee that we are getting infinitely many different fractions $a=\frac{p}{q}$.

3. (a) Assume that (f,g) is a Spanish Couple on $\mathbb{N}$. If $g(a)>g(b)$ then $a>b$ ($a\leq b$ would yield $g(a)\leq g(b)$). Let us introduce the notation

$$g^k(x)=\underbrace{g(g(\cdots g}_{k}(x)\cdots)).$$

   If we assume that $g(x)<x$ for some $x\in\mathbb{N}$ we get $g(g(x))<g(x)<x$, and by induction $(g^k(x))_{k=1}^{\infty}$ is infinite decreasing sequence from $\mathbb{N}$ which is impossible. Hence $g(x)\geq x$ for all $x\in\mathbb{N}$. The same holds for f. If for some

$x \in \mathbb{N}$ we had $f(x) \leq g(x)$, then $g(f(x)) \leq g(g(x)) \leq f(g(g(x))$ which contradicts (ii). Therefore $f(x) > g(x)$ for all $x \in \mathbb{N}$. From $g(f(x)) > f(g^2(x)) > g(g^2(x))$ we conclude that $f(x) > g^2(x)$ and now easy induction implies $f(x) > g^n(x)$ for all $n \in \mathbb{N}$. This is impossible if $(g^n(x))_{n=1}^{\infty}$ is infinite increasing sequence. Hence $g(x) = x$ for all x. This can't be true either because of (ii). This proves that there is no Spanish Couple on $\mathbb{N}$.

(b) The functions $f\left(a - \frac{1}{b}\right) = 3a - \frac{1}{b}$ and $g(a,b) = a - \frac{1}{a+b}$ form a Spanish Couple on the given set S.

4. Using the given properties we get $f(2) = f(2^0 + 0) = f(2^0 - 3) - f(0) = -1$, $f(3) = f(2+1) = f(2-2) - f(1) = 2$, $f(4) = -2$. For every $i \in \{1,2,3\}$ we have $f(2^n - i) = f(2^{n-1} + 2^{n-1} - i) = f(2^{n-1} - t(2^{n-1} - i)) - f(2^{n-1} - i)$. If $2 \mid n$ then $2^{n-1} - i \equiv 2 - i \equiv -(i+1) \pmod 3$, and if $2 \nmid n$ then $2^{n-1} - i \equiv -(i-1) \pmod 3$. Denote $a_k^i = f(2^{2k} - i)$ and $b_k^i = f(2^{2k+1} - i)$. From the previous calculation we get $a_k^i = b_{k-1}^{i+1} - b_{k-1}^i$ and $b_k^i = a_k^{i-1} - a_k^i$. This further implies that $a_k^i = 2a_{k-1}^i - a_{k-1}^{i-1} - a_{k-1}^{i+1}$. By induction we now obtain $a_k^1 = 2 \cdot 3^{k-1}$, $a_k^2 = a_k^3 = -3^{k-1}$. This further gives $b_k^1 = a_k^3 - a_k^1 = -3^k$, $b_k^2 = 3^k$, $b_k^3 = 0$. Moreover, for each $n \in \mathbb{N}$ and each $i \in \{1,2,3\}$: $f(2^n - i) > 0$ if and only if $3 \mid 2^n - i$. If $3 \mid 2^n - i$ then $f(2^n - i) \geq 3^{n/2}$. In addition, $|f(2^n - i)| \leq 2 \cdot 3^{(n-2)/2}$.
Assume that $p \geq 1$ is an integer. Let $\alpha_1, \dots, \alpha_l$ be positive integers such that $3p = 2^{\alpha_1} + \cdots + 2^{\alpha_l}$. In order to prove $f(3p) \geq 0$ let us start with

$$\begin{aligned} f(3p) = \, & f\left(2^{\alpha_1} - t\left(2^{\alpha_2} + \cdots + 2^{\alpha_l}\right)\right) - f\left(2^{\alpha_2} - t\left(2^{\alpha_3} + \cdots + 2^{\alpha_l}\right)\right) \\ & + f\left(2^{\alpha_3} + \cdots + 2^{\alpha_l}\right). \end{aligned}$$

Since $x + y \equiv x - t(y) \pmod 3$ the first term on the RHS is $\geq 3^{\alpha_1/2}$ while $f(2^{\alpha_2} - t(2^{\alpha_3} + \cdots + 2^{\alpha_l})) \leq 0$. It suffices to establish $f(2^{\alpha_3} + \cdots + 2^{\alpha_l}) \leq 3^{\alpha_1/2}$. Let us prove the following: $|f(m)| \leq 3^{n/2}$ for all m, n with $m < 2^n$.
The last statement is true for $n = 1,2$. Assume that the statement holds for some n and assume that $m < 2^{n+1}$. If $m < 2^n$ we are done. Otherwise, let $m = 2^n + j$, for some $0 \leq j < 2^n$. $|f(m)| = |f(2^n - t(j)) - f(j)| \leq |f(2^n - t(j)| + |f(j)|$. Since $|f(2^n - t(j))| \leq 2 \cdot 3^{(n-2)/2}$ and $|f(j)| \leq 3^{n/2}$ we get $|f(m)| \leq 3^{(n+1)/2}$.

5. Using the inequality between arithmetic and geometric mean we get:

$$\begin{aligned} & 3\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) + \left(\frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}\right) \\ = & \left(2\frac{a}{b} + \frac{b}{c} + \frac{a}{d}\right) + \left(2\frac{b}{c} + \frac{c}{d} + \frac{b}{a}\right) + \left(2\frac{c}{d} + \frac{d}{a} + \frac{c}{b}\right) + \left(2\frac{d}{a} + \frac{a}{b} + \frac{d}{c}\right) \\ \geq & \, 4\sqrt[4]{\frac{a^3}{bcd}} + 4\sqrt[4]{\frac{b^3}{cda}} + 4\sqrt[4]{\frac{c^3}{dab}} + 4\sqrt[4]{\frac{d^3}{abc}} = 4(a+b+c+d) \\ > & \, 3\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) + (a+b+c+d). \end{aligned}$$

The required inequality now follows immediately.

6. Assume the contrary, that f is onto. There exists a sequence of numbers a_n such that $f(a_n) = n$.
If $f(u) = f(v)$ then $f(u+1/n) = f(a_n+1/f(u)) = f(a_n+1/f(v)) = f(v+1/n)$ for all $n \in \mathbb{N}$ and by induction $f(u+m/n) = f(v+m/n)$ for all $m,n \in \mathbb{N}$.
Applying this to $u = a_1$, $v = a_1 + 1/f(a_1 - 1)$, and $n = f(a_1 - 1)$ we get $1 = f(a_1) = f(a_1 + 1/n) = f(a_1 + 2/n) = \cdots = f(a_1 + 1)$. This further implies that $f(a_1 + q + 1) = f(a_1 + q)$ for all rational numbers q.
For fixed y we have

$$\begin{aligned}\Gamma(y) &= \left\{ f\left(y + \frac{1}{n}\right) : n \in \mathbb{N} \right\} = \left\{ f\left(y + \frac{1}{f(x)}\right) : x \in \mathbb{R} \right\} \\ &= \left\{ f\left(x + \frac{1}{f(y)}\right) : x \in \mathbb{R} \right\} = \mathbb{N}.\end{aligned}$$

Particularly, $\Gamma(a_1) = \mathbb{N}$ so we could assume that $a_2, a_3, \ldots$ are chosen from $\Gamma(a_1)$. Hence for each $n \geq 2$ there exists $k_n \in \mathbb{N}$ such that $a_n = a_1 + 1/k_n$. Now we have $f(a_1 + 1/n) = f(a_1 + 1/f(a_n)) = f(a_n + 1) = f(a_1 + \frac{1}{k_n} + 1) = f(a_1 + 1/k_n) = f(a_n) = n$.
Since $\Gamma(a_1 + \frac{1}{3}) = \mathbb{N}$ there exists d such that $f(a_1 + \frac{1}{3} + \frac{1}{d}) = 1$. Assume that $\frac{1}{3} + \frac{1}{d} = \frac{p}{q}$ for relatively prime numbers p and q. Since $\frac{1}{3} + \frac{1}{d} \neq 1$ we have $q > 1$. Let k be an integer such that $kp \equiv 1 \pmod{q}$. Then $1 = f(a_1 + p/q) = f(a_1 + kp/q) = f(a_1 + 1/q) = q$ which is a contradiction.

7. The left-hand side can be rewritten as

$$\begin{aligned}L = &\frac{(a-c)^2}{a+b+c} + \frac{(b-d)^2}{b+c+d} + \\ &(a-c)(b-d) \cdot \left(\frac{2d+b}{(b+c+d)(d+a+b)} - \frac{2c+a}{(a+b+c)(c+d+a)} \right).\end{aligned}$$

In order to prove that $L \geq 0$ it suffices to establish the following inequality:

$$\begin{aligned}&(a-c)(b-d) \left(\frac{2d+b}{(b+c+d)(d+a+b)} - \frac{2c+a}{(a+b+c)(c+d+a)} \right) \\ \geq\ &\frac{-2\,|(a-c)(b-d)|}{\sqrt{a+b+c} \cdot \sqrt{b+c+d}}.\end{aligned}$$

If $a = c$ or $b = d$ the inequality is obvious. Assume that $a > c$ and $b > d$. Our goal is to prove that

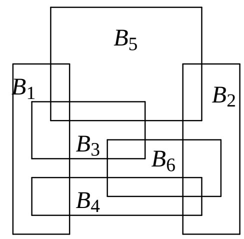
$$\left| \frac{2d+b}{(b+c+d)(d+a+b)} - \frac{2c+a}{(a+b+c)(c+d+a)} \right| \leq \frac{2}{\sqrt{a+b+c} \cdot \sqrt{b+c+d}}.$$

Both fractions on the left-hand side are positive, hence it is enough to prove that each of them is smaller than the right-hand side. These two inequalities are analogous, so let us prove the first one. After squaring both sides, cross-multiplying, and subtracting we get:

$$-(2d+b)^2(a+b+c)+4(d+a+b)^2(b+c+d)$$
$$=-(2d+b)^2(a+b+c)+[(2d+b)+(2a+b)]^2(b+c+d)$$
$$=-(2d+b)^2\cdot a+(2d+b)^2\cdot d+(2a+b)^2(b+c+d)+$$
$$2\cdot(2d+b)(2a+b)(b+c+d)$$
$$>-(2d+b)^2\cdot a+(2d+b)(2a+b)(2b+2c+2d)$$
$$>-(2d+b)^2\cdot a+(2d+b)^2(2a+b)>0.$$

The equality holds if and only if $a=c$ and $b=d$.

8. The largest such n is equal to 6. Six boxes can be placed in a plane as shown in the picture.



Let us now prove that $n \leq 6$. Denote by X_i and Y_i the projections of the box B_i to the lines Ox and Oy respectively. If $i \not\equiv j \pm 1 \pmod{n}$ then $X_i \cap X_j \neq \emptyset$ and $Y_i \cap Y_j \neq \emptyset$. If $i \equiv j \pm 1 \pmod{n}$ then $X_i \cap X_j = \emptyset$ or $Y_i \cap Y_j = \emptyset$.
We will now prove that there are at most 3 values for i such that $X_i \cap X_{i+1} = \emptyset$. Assume that $X_1 = [a_1, b_1]$, $\dots$, $X_n = [a_n, b_n]$. Without loss of generality we may assume that $a_1 = \max\{a_1, a_2, \dots, a_n\}$. If $b_i < a_1$ for some $i \in \{2, 3, \dots, n\}$ then $X_i \cap X_1 = \emptyset$ hence $i \in \{2, n\}$. Therefore $a_1 \in X_3 \cap X_4 \cap \cdots \cap X_{n-1}$ and $X_2 \cap X_3$, $X_n \cap X_{n-1}$, $X_1 \cap X_2$, and $X_1 \cap X_n$ are the only possible intersections that could be empty. We will prove that not all of these sets can be empty. Assume the contrary. Then $a_3 \in (b_2, a_1)$, $b_n \in (a_3, a_1)$, and $a_{n-1} \in (b_n, a_1)$. This implies that $a_{n-1} > b_2$ and $X_{n-1} \cap X_2 = \emptyset$, a contradiction.
In the similar way we prove that at most three of the intersections $Y_1 \cap Y_2$, $\dots$, $Y_n \cap Y_1$ are empty. However there are n empty sets among the intersections $X_1 \cap X_2$, $X_2 \cap X_3$, $\dots$, $X_n \cap X_1$, $Y_1 \cap Y_2$, $Y_2 \cap Y_3$, $\dots$, $Y_n \cap Y_1$ yielding $n \leq 3 + 3 = 6$.

9. Let us call a permutation *nice* if it satisfies the stated property. We want to calculate the number x_n of nice permutations. For $n = 1, 2, 3$ every permutation is nice hence $x_n = n!$ for $n \leq 3$.
Assume now that $n \geq 4$. From $(n-1) \mid 2(a_1 + \cdots + a_{n-1}) = 2[(1 + \cdots + n) - a_n] = n(n+1) - 2a_n = (n+2)(n-1) - 2(a_n - 1)$ we conclude that $(n-1) \mid 2(a_n - 1)$. If n is even we immediately conclude that $a_n = n$ or $a_n = 1$.
Let us prove that $a_n \in \{1, n\}$ for odd n. Assume the contrary. Then $n - 1 = 2(a_n - 1)$, i.e. $a_n = \frac{n+1}{2}$. Then $n - 2 \mid 2(a_1 + \cdots + a_{n-2}) = n(n+1) - 2a_n - 2a_{n-1} = n(n+1) - (n+1) - 2a_{n-1} = (n-2)(n+2) + 3 - 2a_{n-1}$ which gives $n - 2 \mid 2a_{n-1} - 3$. Since $\frac{2a_{n-1}-3}{n-2} \leq \frac{2n-3}{n-2} = 1 + \frac{1}{n-2} < 2$ we get $n - 2 = 2a_{n-1} - 3$. This implies that $a_{n-1} = \frac{n+1}{2} = a_n$ which is a contradiction.
Therefore, for $n \geq 4$ we must have $a_n \in \{1, n\}$. There are x_{n-1} nice permutations for $a_n = n$, and for $a_n = 1$, the problem reduces to counting the nice permutations of the set $\{2, 3, \dots, n\}$ satisfying the given property. However, since $2(a_1 + \cdots + a_k) = 2k + 2((a_1 - 1) + \cdots (a_k - 1))$ we get $k \mid 2(a_1 + \cdots + a_k)$ if and

only if $k \mid 2[(a_1-1)+\cdots+(a_k-1)]$. This provides a bijection between the nice permutations of $\{2,3,\ldots,n\}$ and the nice permutations of $\{1,2,\ldots,n-1\}$. Thus we have $x_n = 2x_{n-1}$ for $n \geq 4$, which implies $x_n = 2^{n-3}\cdot x_3 = 6\cdot 2^{n-3}$ for $n \geq 4$.

10. Since the area of a triangle ABC is equal to $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ we have that $(0,0)$ and (a,b) are k-friends if and only if there exists a point (x,y) such that $ay-bx=\pm 2k$. According to the Euclid's algorithm such integers x and y will exist if and only if $\gcd(a,b) \mid 2k$. Similarly (a,b) and (c,d) are k-friends if and only if $\gcd(c-a,d-b) \mid 2k$.
Assume that there exists a k-clique S of size n^2+1 for some $n \geq 1$. Then there are two elements (a,b), $(c,d) \in S$ such that $a \equiv c \pmod n$ and $b \equiv d \pmod n$. This implies $n \mid \gcd(a-c,b-d) \mid 2k$, or equivalently, $n \mid 2k$.
Therefore, for a k-clique of size 200 to exist, we must have $n \mid 2k$ for all $n \in \{1,2,\ldots,14\}$. Therefore $k \geq 4\cdot 9\cdot 5\cdot 7\cdot 11\cdot 13 = 180180$.
It is easy to see that all lattice points from the square $[0,14]^2$ are 180180-friends.

11. The number of sequences in which the lamp i is switched (on or off) exactly α_i times ($i=1,2,\ldots,2n$) is equal to $\frac{k!}{\alpha_1!\cdot\alpha_2!\cdots\alpha_{2n}!}$. Therefore

$$M = k!\cdot\sum\left\{\frac{1}{\alpha_1!\cdots\alpha_n!} : \alpha_1+\cdots+\alpha_n = k, 2\nmid\alpha_1,\ldots,2\nmid\alpha_n\right\}. \qquad (1)$$

Similarly we get

$$N = k!\cdot\sum\frac{1}{\alpha_1!\cdots\alpha_n!\cdot\beta_1!\cdots\beta_n!},$$

where the summation is over all possible $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$ that satisfy $\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n = k$, $2\nmid\alpha_1$, …, $2\nmid\alpha_n$, $2\mid\beta_1$, …, $2\mid\beta_n$. We see that the sum in (1) is equal to the coefficient of X^k in the expansion

$$f(X) = \left(X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots\right)^n = \sinh^n(X)$$

while the sum in (2) is equal to the coefficient of X^k in the expansion

$$\begin{aligned} g(X) &= \left(X+\frac{X^3}{3!}+\frac{X^5}{5!}+\cdots\right)^n\cdot\left(1+\frac{X^2}{2!}+\frac{X^4}{4!}+\cdots\right)^n \\ &= \sinh^n(X)\cdot\cosh^n(X) = \frac{1}{2^n}\sinh^n(2X). \end{aligned}$$

Assume that $\sinh^n(X) = \sum_{i=0}^{\infty} a_i X^i$ for some real numbers a_1, a_2, …. Then we have $\sinh^n(2X) = \sum_{i=0}^{\infty} a_i(2X)^i = \sum_{i=0}^{\infty} a_i\cdot 2^i\cdot X^i$. Therefore

$$\frac{N}{M} = \frac{k!\cdot a_k}{\frac{1}{2^n}\cdot k!\cdot a_k\cdot 2^k} = 2^{n-k}.$$

12. Let m be the average of all elements from S, i.e. $m = \frac{1}{k+l}(x_1+x_2+\cdots+x_{k+l})$. Denote $\Gamma(A) = \frac{1}{k}\sum_{x\in A} x$. Set A is nice if and only if $|\Gamma(A)-m| \leq \frac{1}{2k}$. For each

permutation $\pi = (\pi_1, \dots, \pi_{k+l})$ of S consider the sets A_1^π, A_2^π, $\dots$, A_{k+l}^π defined as $A_i^\pi = \{\pi_i, \pi_{i+1}, \dots, \pi_{i+k-1}\}$ (indeces are modulo $k+l$). We will prove that at least two of the sets A_i^π are nice. Let us paint the sets A_i^π in red and green in the following way: A_i^π is green if and only if $\Gamma(A_i^\pi) \geq m$. Notice that $\Gamma(A_i^\pi) - \Gamma(A_{i+1}^\pi) = \frac{1}{k}(\pi_i - \pi_{i+k})$ is of absolute value $\leq \frac{1}{k}$. Therefore $|\Gamma(A_i^\pi) - m| \leq \frac{1}{2k}$ or $|\Gamma(A_{i+1}^\pi) - m| \leq \frac{1}{2k}$. Hence whenever two consecutive sets in the sequence $A_1^\pi, \dots, A_{k+l}^\pi$ are of different colors one of them must be nice. If there are ≥ 2 sets of each of the colors, it is obvious that at least two of the sets will be nice. Assume that there is only one red set and that it is the only nice set. Without loss of generality assume that A_1^π is red. Then $m(k+l) = \Gamma(A_1^\pi) + \Gamma(A_2^\pi) + \cdots + \Gamma(A_{k+l}^\pi) > (m - \frac{1}{2k}) + m + \frac{1}{2k} + \cdots + m + \frac{1}{2k} \geq (k+l)m$ which is a contradiction. Now we can prove the required statement. To each of $(k+l)!$ permutations of S we assign at least two nice sets. Each set is counted $(k+l) \cdot k! \cdot l!$ times so there are at least $\frac{2(k+l)!}{k+l} \cdot \frac{1}{k! \cdot l!}$ nice sets.

13. We will prove a stronger result, that for each k we have

$$\sum_{i=1}^{k} (|S_i| - (n+1)) \leq (2n+1) \cdot 2^{n-1}. \tag{1}$$

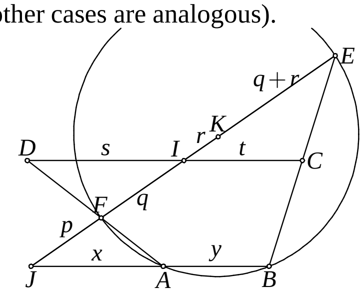
The desired statement follows from (1) because if $|S_i| \geq 2n+2$ for each i then $(2n+1) \cdot 2^{n-1} \geq 2^n \cdot (n+1)$ which is impossible.
We will use induction on n to prove (1). First, for $n = 1$ and any subsets $S_1, S_2, \dots, S_k$ of $\{1,2,3,4\}$ we want to prove $|S_1| + \cdots + |S_k| \leq 2^0 \cdot 3 + k(1+1) = 3 + 2k$. It suffices to verify this only when $|S_i| \geq 3$ for each i. If there is one set with four elements, then $k = 1$ and the inequality is satisfied. If all sets have cardinality 3, then $k \leq 3$ ($\{1,3,4\}$ and $\{2,3,4\}$ can't be both among the chosen sets), hence $3k \leq 6 + 2k$.
Assume now that the statement is true for $n-1$. Let us divide the subsets $S_1, \dots, S_k$ of $\{1,2,\dots,2^{n+1}\}$ in two families: $\mathscr{A} = \{A_1, \dots, A_l\}$ – those with all elements greater than 2^n, and $\mathscr{B} = \{B_1, \dots, B_{k-l}\}$ – the remaining subsets. Using the inductional hypothesis we obtain $\sum_{i=1}^{l}(|A_i| - (n+1)) \leq (2n-1) \cdot 2^{n-2} - l$. Let us denote by α_i the smallest element of $B_i \cap \{2^n+1, \dots, 2^{n+1}\}$ if it exists. Let $H_i = B_i \cap \{1, \dots, 2^n\}$ and $G_i = B_i \cap \{2^n+1, \dots, 2^{n+1}\} \setminus \{\alpha_i\}$. If $i < j$ we claim that $G_i \cap G_j = \emptyset$. If not, considering $z \in G_i \cap G_j$ and taking $y = \alpha_i$, $x \in H_j$ we get a contradiction since $x < y < z$ and $x, z \in B_j$, $y, z \in B_i$.
Sets G_i are disjoint, and the inductional hypothesis holds for sets H_i, hence $\sum_{i=1}^{k-l}(|B_i| - (n+1)) = \sum_{i=1}^{k-l}(|H_i| - n) + \sum_{i=1}^{k-l}|G_i| \leq (2n-1)2^{n-2} + 2^n$. Thus

$$\sum_{i=1}^{k}(|S_i| - (n+1)) \leq (2n-1) \cdot 2^{n-2} \cdot 2 - l + 2^n \leq (2n+1)2^{n-1}.$$

14. Let A', B', C' be the midpoints of BC, CA, AB, respectively, and A'', B'', C'' be the midpoints of HA, HB, HC respectively. Let O be the circumcenter of $\triangle ABC$ and R its circumradius. Pythagoras theorem implies $OA_1^2 = OA'^2 + A'A_1^2 =$

$OA'^2 + A'H^2$. Since $HA'OA''$ is a parallelogram we have that $OA'^2 + A'H^2 = \frac{1}{2}(OH^2 + A'A''^2)$. However, since $A''A'OA$ is a parallelogram we have that $A'A'' = OA = R$. Thus $OA_1^2 = \frac{1}{2}(R^2 + OH^2)$. Similar relations for OA_2, OB_1, OB_2, OC_1, OC_2 imply that the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle with center O.

15. Assume that the distribution of the points is such that $ABEF$ is a convex quadrilateral and C belongs to the segment BE (other cases are analogous).
Let $JF = p$, $FI = q$, $IK = r$. Then $KE = q + r$. Let us further denote $DI = s$, $IC = t$, $JA = x$, $AB = y$. Since $ABEF$ is cyclic we have $JA \cdot JB = JF \cdot JE$, i.e. $x(x+y) = p(p+2q+2r)$. From $CD \| JB$ we have $\frac{s}{x} = \frac{q}{p}$ and $\frac{t}{x+y} = \frac{q+2r}{p+2q+2r}$. The last three equalities imply that $st = q(q+2r)$.



The quadrilateral $ABKI$ is cyclic if and only if $x(x+y) = (p+q)(p+q+r)$. $JCKD$ is cyclic if and only if $(p+q)r = st$. We want to prove that

$$x(x+y) = (p+q)(p+q+r) \;\Leftrightarrow\; (p+q)r = st.$$

Using the equalities we already have, we can eliminate x, $x+y$, s, and t from the previous equivalence. Hence it suffices to prove that:

$$\left[\frac{p(p+2q+2r)}{(p+q)(p+q+r)} = 1 \;\Leftrightarrow\; \frac{(p+q)r}{q(q+2r)} = 1\right]$$
$$\Leftrightarrow [p(p+2q+2r) = (p+q)(p+q+r) \;\Leftrightarrow\; (p+q)r = q(q+2r)].$$

The last equivalence becomes obvious once we multiply all the terms.

16. Let us first consider the case $EQ \neq EP$. Assume that $EQ < EP$ and denote by A' and D' the intersections of EA and ED with the circumcircle of $APQD$. Then $\angle PAA' = \angle PAD' + \angle D'AA' = \angle PAD' + \angle D'DA'$ while $\angle QDD' = \angle QDA' + \angle A'DD'$ hence $\angle QDA' = \angle PAD'$. This means that $A'Q = PD'$ and $A'D' \| QP$. Therefore $\angle DEQ = \angle DD'A' = \angle DAA'$ hence QE is a tangent to the circumcircle of $\triangle DAE$. Let M be the intersection of AD and PQ . Then $ME^2 = MD \cdot MA$. Since $APQD$ is cyclic we have that $MD \cdot MA = MQ \cdot MP$ hence $ME^2 = MQ \cdot MP$. Assume that BC intersects PQ at a point N. Then $NE^2 = NQ \cdot NP$, and since there is the unique point X on the line PQ for which $XE^2 = XQ \cdot XP$ we conclude that $M \equiv N$. Now from $MD \cdot MA = ME^2 = MC \cdot MB$ we get that $ABCD$ is cyclic.
If $EQ = EP$ then it is easy to prove that the perpendicular bisectors of AD, BC, PQ coincide hence $ABCD$ is an isosceles trapezoid hence it is cyclic.

17. Let M be the intersection point of QF and PE. We need to prove that $\angle QMP = \angle BAC$. Since $\angle MQP = \angle QAB$ (QB is a tangent to the circle around $\triangle QFA$) it is enough to prove that $\angle QAB + \angle BAC = \angle QMP + \angle MQP$, or, equivalently $\angle QAE = \angle EPC$. Therefore we need to prove that $AQPE$ is a cyclic quadrilateral. From $BQ^2 = BF \cdot BA = BP^2$ we get $BP = BQ$. Adding $BF \cdot BA = BP^2$ to $AF \cdot$

$AB = AE \cdot AC$ (which holds since $BCEF$ is cyclic) we get $AB^2 = AE \cdot AC + BP^2$. From Pythagoras theorem we have $AB^2 = AE^2 + BE^2 = AE^2 + BC^2 - CE^2$ we get $BC^2 - CE^2 = AE \cdot EC + BP^2$. This implies that $BC^2 - BP^2 = CE^2 + AE \cdot EC$, or equivalently

$$CE \cdot (CE + AE) = (BC + BP)(BC - BP) = CQ \cdot CP.$$

Thus $CE \cdot CA = CP \cdot CQ$ and $QPEA$ is cyclic.

18. We will use the induction on k. The statement is valid for $k = 0$ as there is at least one point P for which (OP) doesn't intersect any of the lines from L.
Assume that the statement holds for $k-1$. Consider the point O and the line (or one of the lines if there are more) whose distance from O is the smallest. Denote this line by l. That line contains $n-1$ points from I. We will first prove that there are at least $k+1$ red points on l. We start by noticing that there exists a point $P \in l \cap I$ such that (OP) doesn't intersect any of the lines from L. P divides the line l in two rays – assume that one of them contains the points $P_1, P_2, \dots, P_u \in I$, while the other ray contains the points $Q_1, \dots, Q_{n-2-u} \in I$. Assume that P_is are sorted according to their distance from P, and the same holds for Q_is. Consider the open segments (OP_i) and (OP_{i+1}). Each line not containing any of P_i and P_{i+1} must intersect either both or none of these segments. The line passing through P_i (other than l) could intersect (OP_{i+1}) and similar fact holds for the line passing through P_{i+1}. Hence the number of intersections of (OP_i) and (OP_{i+1}) with lines from L differ by at most 1. Therefore P_1, P_2, $\dots$, $P_{\min\{k,u\}}$ are all red. Similar holds for Q_is hence there are at least $k+1$ red points on l.
If we remove l together with $n-1$ points on it, the remaining configuration allows us to apply the inductional hypothesis. There are at least $\frac{1}{2}k \cdot (k+1)$ points G from $I \setminus \{l\}$ for which (OG) intersects at most $k-1$ lines from $L \setminus \{l\}$. Therefore there are at least $\frac{1}{2}k \cdot (k+1) + k + 1 = \frac{1}{2}(k+1)(k+2)$ red points.

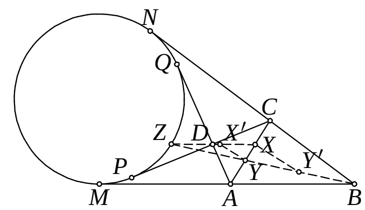
19. Assume first that there exists a point P inside $ABCD$ with the described property. Let K, L, M, N be the feet of perpendiculars from P to AB, BC, CD, and DA respectively. We have $\angle KNM = \angle KNP + \angle PNM = \angle KAP + \angle PDM = 90^\circ$ and similarly $\angle NKL = \angle KLM = \angle LMN = 90^\circ$ hence $KLMN$ is a rectangle. Denote by W, X, Y, Z the feet of perpendiculars from P to KL, LM, MN, and NK. From $\triangle PLX \sim \triangle PCM$ we get that $CM = \frac{XL}{PX} \cdot PM = PM \cdot \frac{PW}{PX}$. Similarly $DM = PM \cdot \frac{PZ}{PY}$, $CL = PL \cdot \frac{PY}{PX}$, $BL = PL \cdot \frac{PZ}{PW}$. Notice that

$$CM : DM = \frac{PW \cdot PY}{PZ \cdot PX} = CL : BL$$

hence $BD \| ML$. Similarly $AC \| LK$ hence $AC \perp BD$.
Conversely, assume that $ABCD$ is a convex quadrilateral for which $AC \perp BD$. Let P' be any point in the plane and consider a triangle $M'P'L'$ for which $M'L' \| BD$, $P'M' \perp CD$, and $P'L' \perp CB$. Let K' be the point for which $P'K' \perp AB$ and $K'L' \| AC$. Let N' be the point such that $K'L'M'N'$ is a rectangle. Consider the four lines α_k, α_l, α_m, α_n through K', L', M', N' perpendicular to $P'K'$, $P'L'$, $P'M'$, and

$P'N'$ respectively. Let $A' = \alpha_k \cap \alpha_n$, $B' = \alpha_k \cap \alpha_l$, $C' = \alpha_l \cap \alpha_m$, and $D' = \alpha_m \cap \alpha_n$. Using the previously established result we have: $A'C' \| K'L'$ and $B'D' \| M'L'$. We also have $C'D' \| CD$, $B'C' \| BC$, $A'B' \| AB$ hence $\triangle DCB \sim \triangle D'C'B'$ and $\triangle ABC \sim \triangle A'B'C'$. Thus there exists a homothety that takes $A'B'C'D'$ to $ABCD$ and this homothethy will map P' into the point P with the required properties.

20. Let M, N, P, Q be the points of tangency of ω with AB, BC, CD, and DA, respectively. We have that $AB + AD = AB + AQ - QD = AB + AM - DP = BM - CP + CD = BN - CN + CD = BC + CD$. Denote by X and Y the points of tangency of ω_1 and ω_2 with AC. Then we have $AB = AX + BC - CX$ and $AD = AY + CD - CY$.



Together with $AB + AD = BC + CD$ this yields to $AX - CX = CY - AY$. Since $AX + CX = CY + AY$ we conclude that $AX = CY$ hence Y is the point of tangency of AC and the excircle ω_B of $\triangle ABC$ that corresponds to B. Similarly, the excircle ω_D corresponding to D of $\triangle ADC$ passes through X.
Consider the homothety that maps ω_B to ω. Denote by Z the image of Y under this homothety. Z belongs to the tangent of ω that is parallel to AC. Therefore Z is the image of X under the homothety with center D that maps ω_D to ω. Denote by X' and Y' the intersections of DX and BY with ω_2 and ω_1 respectively. Circles ω_1 and ω_B are homothetic with center B, hence Y' the image of Y under this homothety. Moreover, Y' belongs to the tangent of ω_1 that is parallel to AC. This implies that XY' is a diameter of ω_1. Similarly, $X'Y$ is a diameter of ω_2. This implies that $X'Y \| XY'$ which means that $\triangle ZX'Y \sim \triangle ZXY'$ and Z is the center of homothety that maps ω_2 to ω_1. This finishes the proof of the required statement.

21. Assume the contrary. If two of the numbers are the same then so are all three of them. Let us therefore assume that all of a, b, c are different. The given conditions imply that
$$\frac{a^n - b^n}{a-b} \cdot \frac{b^n - c^n}{b-c} \cdot \frac{c^n - a^n}{c-a} = -p^3,$$
which immediately implies that some of the numbers a, b, c have to be negative. Moreover, n can't be odd since otherwise each of the fractions would be positive. Assume first that p is odd. Since $2 \nmid \frac{a^n-b^n}{a-b} = a^{n-1} + a^{n-2}b + \cdots + b^{n-1}$ the numbers a and b have to be of different parity. Similarly, $2 \nmid b - c$ and $2 \nmid c - a$ which is not possible.
We are left with the case $p = 2$. Writing $n = 2m$ we derive $(a^m + b^m) \cdot (b^m + c^m) \cdot (c^m + a^m) \cdot \frac{a^m-b^m}{a-b} \cdot \frac{b^m-c^m}{b-c} \cdot \frac{c^m-a^m}{c-a} = -8$. This means that $a^m + b^m = \pm 2$, $a^m - b^m = \pm(a - b)$ and analogous equalities hold for the pairs (b,c) and (c,a). If m is even then $|a| = |b| = |c| = 1$ which means that at least two of a, b, c have to be the same.

If m is odd then $\pm 2 = a^m + b^m$ is divisible by $a+b$. Since $a^m + b^m \equiv a+b$ (mod 2) we conclude that $a+b = \pm 2$. Similarly $b+c = \pm 2$ and $c+a = \pm 2$. At least two of a, b, c have to be the same which is a contradiction.
Remark. The statement of the problem remains valid if we replace the assumption that p is prime with the assumption $2 \mid p$ or $p = 2$.

22. Assume the contrary. Without loss of generality we may assume that these numbers are relatively prime (otherwise we could divide them by their common divisor). We may also assume that $a_1 < a_2 < \cdots < a_n$. For each $i \in \{1,2,\dots,n-1\}$ there exists $j \in \{1,2,\dots,n-1\}$ such that $a_n + a_i \mid 3a_j$. This together with $a_n + a_i > a_j$ implies that $a_n + a_i$ is divisible by 3 for all i.
There exists $k \in \{1,2\}$ such that $a_n \equiv k$ (mod 3) and $a_i \equiv 3-k$ (mod 3) for all $i \neq n$. For each $i \in \{1,2,\dots,n-2\}$ there exists j such that $a_{n-1} + a_i \mid 3a_j$. Since $a_{n-1} + a_i$ is not divisible by 3 we must have $a_{n-1} + a_i \mid a_j$ hence $j = n$ and we conclude that $a_{n-1} + a_i \mid a_n$ for all $i \in \{1,2,\dots,n-2\}$. Let $l \in \{1,2,\dots,n\}$ be such an integer for which $a_n + a_{n-1} \mid 3a_l$. Adding the inequalities $a_n + a_{n-1} \le 3a_l$ and $a_{n-1} + a_l \le a_n$ gives that $a_{n-1} \le a_l$ thus either $l = n$ or $l = n-1$.
In the first case $u(a_{n-1} + a_n) = 3a_n$ for some $u \in \mathbb{N}$. We immediately see that $u < 3$ and $u > 1$. Hence $u = 2$ and $2a_{n-1} = a_n$. However, this is impossible since for each $i \in \{1,2,\dots,n-2\}$ the number $a_{n-1} + a_i$ divides $a_n = 2a_{n-1}$.
On the other hand, if $a_{n-1} + a_n \mid 3a_{n-1}$ then there exists $v \in \mathbb{N}$ for which $v(a_{n-1} + a_n) = 3a_{n-1}$. If $v \ge 2$ then $2a_{n-1} + 2a_n \le 3a_{n-1}$ which is impossible. Hence $v = 1$ and we get $a_n = 2a_{n-1}$. In the same way as in the previous case we get a contradiction.

23. We will use the induction on n. Observe that $a_n \ge (a_{n+1}, a_n) > a_{n-1}$. Obviously, $a_0 \ge 1$, and $a_1 \ge a_0 + 1 \ge 2$. From $a_{k+1} - a_k \ge (a_{k+1}, a_k) \ge a_{k-1} + 1$ we get $a_2 \ge 4$ and $a_3 \ge 7$. It is impossible to have $a_3 = 7$ since $(a_3, a_2) > a_1 = 2$ would imply $a_2 = 7 = a_3$. Hence we have that the statement is satisfied for $n \in \{0,1,2,3\}$.
Assume that $n \ge 2$ and $a_i \ge 2^i$ for all $i \in \{0,1,\dots,n\}$. We need to prove that $a_{n+1} \ge 2^{n+1}$. Let us denote $d_n = (a_{n+1}, a_n)$. We have $d_n > a_{n-1}$. Let $a_{n+1} = kd_n$ and $a_n = ld_n$. If $k \ge 4$ we are done because $a_{n+1} \ge 4d_n > 4a_{n-1} \ge 4 \cdot 2^{n-1} = 2^{n+1}$. If $l \ge 3$ then $a_{n+1} > a_n$ implies $k \ge 4$. If $l = 1$ then $a_{n+1} \ge 2a_n \ge 2^{n+1}$.
Hence the only remaining case to consider is $a_n = 3d_n$, $a_{n-1} = 2d_n$. Obviously, $d_{n-1} = (2d_n, a_{n-1}) > a_{n-2}$. If $a_{n-1} = d_{n-1}$ then from $a_{n-1} < d_n$ and $a_{n-1} \mid 2d_n$ we get $\frac{2d_n}{a_{n-1}} \ge 3$ and $d_n \ge \frac{3}{2}a_{n-1} \ge \frac{3}{2} \cdot 2^{n-1}$. Now $a_{n+1} = 3d_n \ge 9 \cdot 2^{n-2} > 2^{n+1}$.
If $a_{n-1} \ge 3d_{n-1}$ then $d_n > a_{n-1} \ge 3d_{n-1}$. Since $d_{n-1} = (2d_n, a_{n-1})$ there exists $s \in \mathbb{N}$ such that $2d_n = sd_{n-1}$. This implies that $d_n > 3 \cdot \frac{2d_n}{s}$ which means $s > 6$, or $s \ge 7$. Therefore $2d_n \ge 7d_{n-1} > 7 \cdot 2^{n-2}$ and $a_{n+1} = 3d_n > 2^{n+1}$.
It remains to consider the case $a_{n-1} = 2d_{n-1}$. From $(2d_n, 2d_{n-1}) = d_{n-1}$ we conclude that $d_n = \frac{d_{n-1}}{2}w$ for some odd integer $w \ge 3$. From $a_{n-1} < d_n$ we get $2d_{n-1} < d_n$ hence $w \ge 5$. If $w \ge 7$ then $a_{n+1} \ge 3 \cdot 7 \cdot \frac{d_{n-1}}{2} > 21 \cdot 2^{n-3} > 2^{n+1}$ hence it remains to consider the case $w = 5$. We now have $2^{n-3} \le a_{n-3} < d_{n-2} = (2d_{n-1}, a_{n-2})$. If $a_{n-2} \ge 2d_{n-2}$ then $2d_{n-1} \ge 3d_{n-2} > 3 \cdot 2^{n-3}$. There-

fore $a_{n+1} = 3 \cdot \frac{5d_{n-1}}{2} \geq 45 \cdot 2^{n-4} > 2^{n+1}$. If $a_{n-2} = d_{n-2}$ then from $a_{n-2} < d_{n-1}$ we get again $2d_{n-1} \geq 3d_{n-2}$ and $a_{n+1} \geq 2^{n+1}$.

24. First we prove that the numbers $\binom{2^n-1}{k}$ are all odd. Let M be the largest integer for whcich 2^M divides $(2^n-1)!$. Then $M = \sum_{i=1}^{n-1}\left[2^{n-i} - \frac{1}{2^i}\right] = \sum_{i=1}^{n-1}\left(2^{n-i}-1\right)$. The largest number N for which 2^N divides $k! \cdot (2^n-1-k)!$ satisfies

$$N = \sum_{i=1}^{n-1}\left(\left[\frac{k}{2^i}\right] + \left[2^{n-i} - \frac{k+1}{2^i}\right]\right).$$

Each summand on the right-hand side is equal to $2^{n-i}-1$ (write $k = q_i \cdot 2^i + r_i$, for $0 \leq r_i < 2^i$). Hence $M = N$ and $\binom{2^n-1}{k}$ is odd.

Let us prove that $\binom{2^n-1}{k}$ give different remainders modulo 2^n. This is valid for $n = 1$. Assume that this holds for some $n > 1$. We claim that the sets $A_i = \left\{\binom{2^{n+1}-1}{2i}, \binom{2^{n+1}-1}{2i+1}\right\}$ and $B_i = \left\{\binom{2^n-1}{i}, 2^{n+1} - \binom{2^n-1}{i}\right\}$ are the same modulo 2^{n+1} for each $i = 0, 1, \dots, 2^{n-1}-1$. We also claim that that all numbers from $\bigcup_{i=0}^{2^{n-1}-1} B_i$ are different modulo 2^{n+1}. These two claims will imply the desired result. Let us show that $\binom{2^{n+1}-1}{2i} \equiv -\binom{2^{n+1}-1}{2i+1} \pmod{2^{n+1}}$ and that one of these two numbers is congruent to $\binom{2^n-1}{i}$. The first congruence follows from

$$\begin{aligned}\binom{2^{n+1}-1}{2i} &= \binom{2^{n+1}}{2i+1} - \binom{2^{n+1}-1}{2i+1} \\ &= \frac{2^{n+1}}{2i+1}\binom{2^{n+1}-1}{2i} - \binom{2^{n+1}-1}{2i+1} \\ &\equiv -\binom{2^{n+1}-1}{2i+1} \pmod{2^{n+1}},\end{aligned}$$

while the second is true because

$$\begin{aligned}\binom{2^{n+1}-1}{2i} &= \prod_{k=0}^{i-1} \frac{2^{n+1}-(2k+1)}{2k+1} \cdot \prod_{k=1}^{i} \frac{2^{n+1}-2k}{2k} \\ &= \prod_{k=0}^{\left[\frac{i-1}{2}\right]} \frac{2^{n+1}-(2k+1)}{2k+1} \cdot \prod_{k=1}^{i} \frac{2^n-k}{k} \\ &\equiv (-1)^i \cdot \binom{2^n-1}{i} \pmod{2^{n+1}}.\end{aligned}$$

It remains to show that $\bigcup_{i=0}^{2^{n-1}-1} B_i$ have all elements different modulo 2^{n+1}. Inductional hypothesis implies that $\binom{2^n-1}{i}$ has different remainder than $\binom{2^n-1}{j}$ for $i \neq j$. The same holds for $2^{n+1} - \binom{2^n-1}{i}$ and $2^{n+1} - \binom{2^n-1}{j}$. From $\binom{2^n-1}{2k} + \binom{2^n-1}{2k+1} \equiv 2^n$ we have that $\binom{2^n-1}{i} \equiv 2^{n+1} - \binom{2^n-1}{j} \equiv 0 \pmod{2^{n+1}}$ if and only if there exists k such that $\{i, j\} = \{2k, 2k+1\}$ for some k. However, in that case $\binom{2^n-1}{i} + \binom{2^n-1}{j} = \frac{2^n}{2k+1}\binom{2^n-1}{2k} \not\equiv 0 \pmod{2^{n+1}}$.

25. If p is a prime number, then $d(f(p))$ has p divisors, and must be a power of a prime. Hence $f(p) = q^{p-1}$ for some prime number q. Let us show that $q = p$. Consider first the case $p > 2$. From $f(2p) \mid (2-1) \cdot p^{2p-1} \cdot f(2)$ and $f(2p) \mid (p-1) \cdot 2^{2p-1} \cdot f(p) = (p-1) \cdot 2^{2p-1} \cdot q^{p-1}$ we conclude that $f(2p) \mid (p^{2p-1} \cdot f(2), (p-1) \cdot 2^{2p-1} \cdot q^{p-1}) = (f(2), (p-1) \cdot 2^{2p-1} \cdot q^{p-1})$. Since $f(2p)$ has $2p$ divisors and $f(2)$ is prime this is a contradiction. We also have $f(2) = 2$. Indeed, this follows from $f(6) \mid 3^{6-1} \cdot f(2)$, $f(6) \mid 2 \cdot 2^{6-1} \cdot 3^{3-1}$, and $d(f(6)) = 6$.
Assume now that $x = p_1^{a_1} \cdots p_n^{a_n}$ is a prime factorization of x with $p_1 < \cdots < p_n$. Let $f(x) = q_1^{b_1} \cdots q_m^{b_m}$. From $d(f(x)) = p_1^{a_1} \cdots p_n^{a_1} = (b_1+1) \cdots (b_m+1)$ we conclude that $b_i \geq p_1 - 1$ for all i. The relation $f(x) \mid (p_1 - 1) \cdot (p_1^{a_1-1} \cdots p_n^{a_n})^{x-1} \cdot f(p_1)$ yields to $q_1, \dots, q_m \in \{p_1, \dots, p_n\}$. Hence for each prime p and each $a \in \mathbb{N}$ there is $b \in \mathbb{N}$ such that $f(p^a) = p^b$. From $p^a = b + 1$ we get $f(p^a) = p^{p^a - 1}$.
Now assume that $x \in \mathbb{N}$. There are integers $a_1, \dots, a_n, b_1, \dots, b_n$ such that $x = p_1^{a_1} \cdots p_n^{a_n}$ and $f(x) = p_1^{b_1} \cdots p_n^{b_n}$. For each $i \in \{1, \dots, n\}$ we have $f(x) \mid (p_i^{a_i} - 1) \cdot (x/p_i^{a_i})^{x-1} \cdot p_i^{p_i^{a_i}-1}$ hence $p_i^{b_i} \mid p_i^{p_i^{a_i}-1}$ which implies $b_i + 1 \leq p_i^{a_i}$. Multiplying this for $i = 1, \dots, n$ we get $d(f(x)) = (b_1+1) \cdots (b_n+1) \leq p_1^{a_1} \cdots p_n^{a_n} = x$. Since $d(f(x)) = x$ we must have $b_i = p_i^{a_i} - 1$ for all i and $f(x) = p_1^{p_1^{a_1}-1} \cdots p_n^{p_n^{a_n}-1}$.
It is easy to verify that function f defined by the previous relation satisfies the required conditions.

26. If p is any prime number of the form $p \equiv 1 \pmod 4$ we know that $\left(\frac{-1}{p}\right) = 1$ and there are exactly two numbers $n, m \in \{0, 1, 2, \dots, p-1\}$ whose square is congruent to -1 modulo p. Since the sum of these two numbers is equal to p, one of them is smaller than $p/2$. Assuming that $n < p/2$ let us denote $k = p - 2n$. It suffices to prove that there exist infinitely many prime numbers p for which $k > \sqrt{2n}$. From $p \mid n^2 + 1 = \frac{p^2 - 2pk + k^2}{4} + 1$ we conclude that $p \mid k^2 + 4$. This implies that $k^2 \geq p - 4$. It suffices to prove that $p - 4 > 2n$, i.e. $4 < p - 2n = k$ for infinitely many values of p. However, this will be satisfied since $k \geq \sqrt{p-4} > 4$ for $p > 20$, and there are infinitely many prime numbers greater than 20 that are congruent to 1 modulo 4.

A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathscr{B}(A,B,C)$, $A-B-C$: indicates the relation of *betweenness*, i.e., that B is between A and C (this automatically means that A,B,C are different collinear points).
- $A = l_1 \cap l_2$: indicates that A is the intersection point of the lines l_1 and l_2.
- AB: line through A and B, segment AB, length of segment AB (depending on context).
- $[AB$: ray starting in A and containing B.
- $(AB$: ray starting in A and containing B, but without the point A.
- (AB): open interval AB, set of points between A and B.
- $[AB]$: closed interval AB, segment AB, $(AB) \cup \{A,B\}$.
- $(AB]$: semiopen interval AB, closed at B and open at A, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- ABC: plane determined by points A,B,C, triangle ABC ($\triangle ABC$) (depending on context).
- $[AB,C$: half-plane consisting of line AB and all points in the plane on the same side of AB as C.
- $(AB,C$: $[AB,C$ without the line AB.

- $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$, $\overrightarrow{a} \cdot \overrightarrow{b}$: scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$.
- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle ABC (unless otherwise indicated).
- $k(O, r)$: circle k with center O and radius r.
- $d(A, p)$: distance from point A to line p.
- $S_{A_1A_2\ldots A_n}$, $[A_1A_2\ldots A_n]$: area of n-gon $A_1A_2\ldots A_n$ (special case for $n = 3$, S_{ABC}: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo n, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo p, p being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in x with integer and real coefficients respectively.
- R^*: the set of nonzero elements of a ring R.
- $R[\alpha]$, $R(\alpha)$, where α is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- X_0: $X \cup \{0\}$ for X such that $0 \notin X$.
- X^+, X^-, $aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to x.
- $\lceil x \rceil$: the smallest integer greater than or equal to x.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points AB, the absolute value of the number x, the number of elements of the set S (respectively).
- (x, y), (m, n), (a, b): (ordered) pair x and y, the greatest common divisor of integers m and n, the open interval between real numbers a and b (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notation and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).

- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

B

Codes of the Countries of Origin

ARG	Argentina	HKG	Hong Kong	POL	Poland
ARM	Armenia	HUN	Hungary	POR	Portugal
AUS	Australia	ICE	Iceland	PRK	Korea, North
AUT	Austria	INA	Indonesia	PUR	Puerto Rico
BEL	Belgium	IND	India	ROM	Romania
BLR	Belarus	IRE	Ireland	RUS	Russia
BRA	Brazil	IRN	Iran	SAF	South Africa
BUL	Bulgaria	ISR	Israel	SER	Serbia
CAN	Canada	ITA	Italy	SIN	Singapore
CHN	China	JAP	Japan	SLO	Slovenia
COL	Colombia	KAZ	Kazakhstan	SMN	Serbia and Montenegro
CRO	Croatia	KOR	Korea, South	SPA	Spain
CUB	Cuba	KUW	Kuwait	SVK	Slovakia
CYP	Cyprus	LAT	Latvia	SWE	Sweden
CZE	Czech Republic	LIT	Lithuania	THA	Thailand
CZS	Czechoslovakia	LUX	Luxembourg	TUN	Tunisia
EST	Estonia	MCD	Macedonia	TUR	Turkey
FIN	Finland	MEX	Mexico	TWN	Taiwan
FRA	France	MON	Mongolia	UKR	Ukraine
FRG	Germany, FR	MOR	Morocco	USA	United States
GBR	United Kingdom	NET	Netherlands	USS	Soviet Union
GDR	Germany, DR	NOR	Norway	UZB	Uzbekistan
GEO	Georgia	NZL	New Zealand	VIE	Vietnam
GER	Germany	PER	Peru	YUG	Yugoslavia
GRE	Greece	PHI	Philippines		