EXCITED DETERMINISTIC WALK IN A RANDOM ENVIRONMENT

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Abstract. Excited deterministic walk in a random environment is a non-Markov integer-valued process $(X_n)_{n=0}^{\infty}$, whose jump at time n depends on the number of visits to the site X_n. The environment can be understood as stacks of cookies on each site of $\mathbb{Z}$. Once all cookies are consumed at a given site, every subsequent visit will result in a walk taking a step according to the direction prescribed by the last consumed cookie. If each site has exactly one cookie, then the walk ends in a loop if it ever visits the same site twice. If the number of cookies per site is increased to two, the walk can visit a site x arbitrarily many times before getting stuck in a loop, which may or may not contain x. Nevertheless the moments of X_n are sub-linear in n and we establish monotonicity results on the environment that imply large deviations.

1. Introduction

The excited deterministic walk in a random environment (EDWRE) in dimension $d \geq 1$ is a discrete time process, $(X_n)_{n=0}^{\infty} : \Omega \to \left(\mathbb{Z}^d\right)^{\{0,1,\dots\}}$. For $L, M \in \mathbb{N}$, the set of environments is

$$\Omega = \Omega(L,M) = \Big\{ \omega \in \left([-L,L]^d\right)^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}^d} : \\ \omega(j,z) = \omega(M-1,z) \text{ for each } j \geq M-1 \text{ and each } z \in \mathbb{Z}^d \Big\},$$

where $[a,b] := \{a, a+1, \dots, b\}$. We imagine Ω as stacks of M *cookies*, $\omega(0,z), \dots, \omega(M-1,z)$, at each site $z \in \mathbb{Z}^d$, each with an arrow pointing to an element of the cube $[-L,L]^d$. We assume that Ω is equipped with the product measure $\mathbb{P} = \mathbb{P}_{L,M}$ such that $\{\omega(j,z) : j \in [0, M-1], z \in \mathbb{Z}^d\}$ are i.i.d. with distribution μ supported on $[-L,L]^d$. Note the abuse of notation here, that $\omega \in \Omega$ is both an element of the set of environments, and a random element (via the identity map) with distribution $\mathbb{P}$. We further assume that $\mu(k) > 0$ for all $k \in [-L,L]^d \setminus \{0\}$.

To define the *excited deterministic walk in a random environment*, first let $L_n(z) = L_n(\omega, z)$ denote the number of times that the walker visited z in the time interval $[0, n-1]$,

$$L_n(z) = \left|\{0 \leq j < n : X_j = z\}\right|,$$

where $|A|$ denotes the cardinality of the set A. For each $\omega \in \Omega$, we define $X_n = X_n(\omega)$ recursively as

$$X_0 = 0,$$
$$X_{n+1} = X_n + \omega(L_n(X_n), X_n).$$

The main result of this paper is the large deviations estimate of the probability that X_n is located at a distance of order $O(n)$ from the origin when $d = 1$.

Theorem 1. *Assume that $d = 1$ and fix $M \geq 3$. There exists a function $\phi : [0, L] \to (-\infty, 0]$ such that for each $\lambda \in [0, L]$*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \geq \lambda n\right) = \phi(\lambda). \tag{1}$$

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Remark 1. *The assumption of an i.i.d. environment can be weakened slightly, and we make this assumption merely for the ease of exposition. For instance, Theorem 1 holds if each 'layer' of cookies has a different distribution. That is, we have M distributions, $\mu_0, \dots, \mu_{M-1}$, such that $\mu_j(k) > 0$ for all $k \in [-L, L] \setminus \{0\}$, $\omega(j, z) \sim \mu_j$ for $j \in [0, M-1]$ and $z \in \mathbb{Z}$, and $\{\omega(j, z) : j \in [0, M-1], z \in \mathbb{Z}\}$ are independent. In this case, the only change required in the proof is to redefine $\mu_{\min}$ appearing near the start of Section 2 as*

$$\mu_{\min} = \min\{\mu_j(k) : k \in [-L, L] \setminus \{0\}, j \in [0, M-1]\}.$$

This generalization more closely resembles the models of excited random walks, which we discuss below.

Remark 2. *The function ϕ is concave on $[0, L]$, with $\phi(0) = 0$ and $\phi(\lambda) < 0$ for $\lambda \in (0, L]$. This is proved in section 6.*

Remark 3. *We expect Theorem 1 to hold when $M = 2$, and can prove that it does when $L \leq 2$. However, our proof for $M \geq 3$ does not work when $M = 2$ and $L \geq 3$. The case $M = 1$ was proved using a different method in* [11].

The model studied in this paper is a generalization of deterministic walk in random environment (DWRE) [11] in the same way as excited random walk (ERW) generalizes random walk by allowing several cookies on each site [3]. The model of DWRE traces its origins to the study of stochastic partial differential equations. The viscosity solutions to random Hamilton-Jacobi and Hamilton-Jacobi-Bellman equations can be represented using variational formulas [1, 7, 17]. The controls in the formulas are solutions to ordinary differential equations or stochastic differential equations in random environments whose discrete analogs are deterministic walks in random environments (DWRE) and random walks in random environments (RWRE), respectively [16].

Large deviations for RWRE and EWR were studied in the past and various results were obtained [12, 15, 20, 21, 22]. The approaches from these papers cannot be applied to DWRE or EDWRE because the latter models do not possess the quenched ellipticity property. Results related to the laws of large numbers for non-elliptic random walks were established in [4]. In the case of RWRE, one can assume ellipticity and use the point of view of the particle to see the process as a Markov chain on a probability space with sufficient compactness to apply the Donsker–Varadhan theory [19, 21]. Large deviations analogous to Theorem 1, but in all dimensions, were proved for DWRE by an analysis of loops [11]. However, this loop analysis is not applicable to large deviations of EDWRE.

The model of excited random walk was first introduced by Benjamini and Wilson [3] in which the nearest-neighbor random walk was perturbed by adding a cookie to each site of $\mathbb{Z}$. In later studies the random walks in random environments were modified by adding multiple cookies to each of the sites and a number of results were established about recurrence, ballisticity, monotonicity, and return times to zero [2, 6, 8, 9, 10, 13]. Some of these excited random walks are known to converge to Brownian motion perturbed at extrema [5]. The above results that study the behavior for large n cannot be generalized for our walks if M is kept fixed. One of the properties of EDWRE is almost sure boundedness. However, it is unknown whether different modes of convergence may occur if the assumption $M < \infty$ is changed to an assumption that each site has a finite (but not uniformly bounded) random number of cookies. For the ERW the finiteness of the expected return time to 0 depends on the average drift per site [10]. The return time to 0 for EDWRE is infinite with positive probability. However, conditioned on the event that the walker returns to 0, it is unknown what the expected lengths of the excursions are.

Large deviations for random walks in random environments were studied in [14]. In the case of random walks in excited random environments very little is known in higher dimensions. The methods are often restricted to nearest-neighbor walks. Our main proof is also restricted to one dimension, however we are allowing our walk to make jumps of sizes bigger than 1.

The case $M = 1$ corresponds to DWRE and Theorem 1 can be obtained in arbitrary dimension d [11]. The main argument of the proof used the fact that once the walk visits a site it has visited before, it will

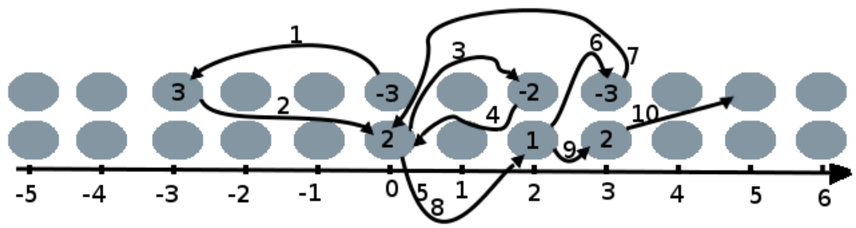

FIGURE 1. The environment described in Example 1.

end in a loop. This can be simply stated as the $0-1-\infty-$*principle*, meaning that in DWRE the number of times a given site can be visited by the walk is zero, one, or infinity. However, we will see in Theorem 2 that EDWRE is a much richer model, and that a site can be visited arbitrarily many times.

The key ingredient in the proof of Theorem 1 is Lemma 4 that establishes a monotonicity property among *favorable environments*. A configuration of cookies on $\mathbb{Z}$ is called a favorable environment if it enables the walk starting at 0 to reach λn in fewer than n steps. Lemma 4 states that for every favorable environment one can change several cookies in $[0, O(\sqrt{n})]$ to make another favorable environment that also allows the walk to avoid any backtrackings over 0. This result was the key to establishing a sub-additivity necessary for proving large deviations.

In the case when the maximal jump size is $L = 2$ one can replace $O(\sqrt{n})$ in Lemma 4 with a finite number. It remains unknown whether $O(\sqrt{n})$ can be replaced by a finite number when $L \geq 3$.

Before delving into properties of the model, it is instructive to consider one concrete example.

Example 1. *Assume that the random environment is created in the following way. Each site of $\mathbb{Z}$ independently choses a sequence of two integers from $\{-3,-2,\ldots,3\}$. In the example depicted by Figure 1, the site 0 has cookies $(-3,2)$, while the site 2 has cookies $(-2,1)$. We will denote the cookies at 0 by $\omega(0,0)=-3$ and $\omega(1,0)=2$. Similarly, $\omega(0,2)=-2$ and $\omega(1,2)=1$.*

If the cookies are as shown in the picture above, then the first 10 steps of the walk are $X_0=0$, $X_1=-3$, $X_2=0$, $X_3=2$, $X_4=0$, $X_5=2$, $X_6=3$, $X_7=0$, $X_8=2$, $X_9=3$, and $X_{10}=5$.

2. Properties of excited walks

The results in this section serve to outline some of the major differences between excited and non-excited walks. In this section we will restrict ourselves to the case $d = 1$. In regular non-excited deterministic walks in random environments, the number of visits to any particular site can be 0, 1, or infinity. The last case corresponds to the situation in which the walk ends in a loop passing through a prescribed number of sites infinitely many times. In an excited environment, the walker may revisit 0, for instance, any number of times $1,2,\ldots,\infty$. However, the probability of revisiting 0 a large finite number of times decays exponentially, as the next theorem demonstrates. For convenience, we let

$$\mu_{\min} \quad = \quad \min\{\mu(k) : k \in [-L,L]\setminus\{0\}\}.$$

Theorem 2. *Assume that $L \geq 2$ and $M \geq 2$. Let D_0 be the cardinality of the set $\{n : X_n = 0\}$. For each $k \in \mathbb{N}$ the following inequality holds*

$$(\mu_{\min})^{4Mk} \leq \mathbb{P}\left(D_0 = k\right) \leq 2\left(1-(\mu_{\min})^{2M+L-2}\right)^{k/2LM}$$

Remark 4. *In the case $L = 1$, $D_0 \in \{1,\ldots,2M-1,\infty\}$. To see this, observe that on the Mth visit to 0, all cookies but the last have been consumed. Assume the last cookie points to the right. The site 0 can be visited at most $M-1$ additional times without being caught in a loop if the top $M-1$ cookies at 1 all point left, and the last cookie points right. It is easy to construct environments that attain each of these values, but it is an interesting problem to compute the distribution of D_0.*

Proof. The lower bound follows from Lemma 1 below.

Lemma 1. *There exist two functions $f, g : \mathbb{Z} \to \{-2,-1,1,2\}$ such that the deterministic sequence x_n defined by $x_0 = 0$ and*

$$x_{n+1} = x_n + \begin{cases} f(x_n), & \text{if } x_n \in \{x_0, \dots, x_{n-1}\}, \\ g(x_n), & \text{if } x_n \notin \{x_0, \dots, x_{n-1}\} \end{cases}$$

contains exactly k terms equal to 0 and has $-2k \le x_n \le 2k-1$ for all n.

Indeed, if we find two such functions, then the event $E \subset \{D_0 = k\}$ can be constructed as follows:

$$E = \{\omega \in \Omega \ : \ \omega(0,z) = g(z) \text{ and } \omega(i,z) = f(z) \text{ for } i \ge 1 \text{ and } -2k \le z \le 2k-1\}.$$

We have $\mathbb{P}(E) \ge (\mu_{\min})^{4Mk} > 0$, so $\mathbb{P}(D_0 = k) \ge (\mu_{\min})^{4Mk}$.

For the upper bound, suppose that V_0^j is the time of the jth visit to 0 (so $V_0^1 = 0$). If $V_0^k < \infty$ and $V_0^{k+1} = \infty$, then the walker cannot get stuck in a loop that includes 0, and the number of visits to 0 must be k. Therefore, between consecutive visits to 0, the walker must see at least one new cookie, otherwise it will be stuck in a loop containing 0. That is, for each $0 \le j \le k-1$, there exists $x \in \left\{X_{V_0^j}, X_{V_0^j+1}, \dots, X_{V_0^{j+1}}\right\}$ such that $L_{V_0^j}(x) \le M-1$. Therefore, by time V_0^k, the walker must have visited at least k/M distinct vertices. Furthermore, this implies that the walker must have visited at least k/LM regions of the form $[iL, (i+1)L-1]$ for $i \in \mathbb{Z}$. That is,

$$\left|\left\{i \in \mathbb{Z} : [iL,(i+1)L-1] \cap \{X_t : 0 \le t \le V_0^k\} \neq \emptyset\right\}\right| \ge \frac{k}{LM}.$$

In order for the walker to revisit 0 at time V_0^k, none of the regions $[iL,(i+1)L-1]$ that the walker visits before this time can be a trap where the walker gets stuck in a loop. An example of a trapping configuration on the interval $[iL,(i+1)L-1]$ has $\omega(j,iL) = 1 = -\omega(j,iL+1)$ for $j \ge 0$ and $\omega(0,iL+x) = -x$ for $x = 2,\dots,L-1$. Therefore, the probability that $[iL,(i+1)L-1]$ is a trapping region is at least $(\mu_{\min})^{2M+L-2}$.

Finally, observe that the set of $i \in \mathbb{Z}$ such that the walker visits $[iL,(i+1)L-1]$ by time V_0^k must be a set of consecutive integers containing 0, since the walker cannot jump over any such region. Therefore, the walker must either visit every such region for $0 \le i \le k/2LM - 1$, or every such region for $-k/2LM + 1 \le i \le 0$. The probability that none of these regions is a trap gives the upper bound. □

Proof of Lemma 1. Let us first define f and g on the set $\mathbb{Z}_-$ of negative numbers, i.e. $\mathbb{Z}_- = \{-1,-2,-3,\dots\}$. If $z \in \mathbb{Z}_-$ is odd we set $f(z) = g(z) = 2$, and if $z \in \mathbb{Z}_-$ is even we set $f(z) = -2$ and $g(z) = 1$.

For $i \in \{0,1,\dots,2k-3\}$ we define

$$g(i) = \begin{cases} -2, & \text{if } i \text{ is even}, \\ -1, & \text{if } i \text{ is odd}; \end{cases} \qquad \text{and} \qquad f(i) = \begin{cases} -2, & \text{if } i \text{ is even}, \\ 2, & \text{if } i \text{ is odd}. \end{cases}$$

We finally define $f(2k-1) = g(2k-1) = -1$ and $f(2k-2) = g(2k-2) = 1$. The values of $f(i)$ and $g(i)$ for $i \ge 2k$ are irrelevant, and can be set to any value. The environment corresponding to g and f is shown in Figure 2.

We will prove that $x_{2i(i+1)} = 0$ for $i \in \{0,1,2,\dots,k-1\}$ and that all other terms of the sequence $(x_n)_{n=0}^{\infty}$ are non-zero.

We will now use induction on i to prove that for each $i \in \{0,1,\dots,k-1\}$ the following holds:

$$\begin{aligned} & x_{2i(i+1)} = 0 \ \text{ and} \\ & \{x_0,\dots,x_{2i(i+1)}\} = \{-2i,-2i+1,\dots,0,1,\dots,2i-1\}. \end{aligned} \tag{2}$$

This is easy to verify for $i = 0$ and $i = 1$. Assume that the statement is true for some i and let us prove it for $i+1$.

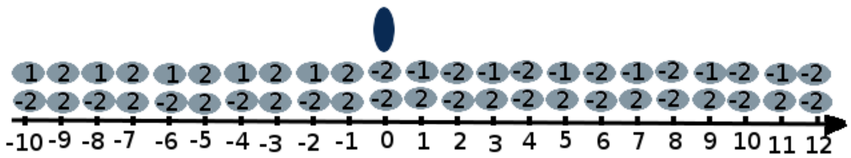

FIGURE 2. Part of the environment constructed in the proof of Lemma 1. The values of the top cookies are given by the function g, while the bottom cookies are given by the function f. In this environment, the walker will return to 0 at least 5 times (after the 5th return, it moves to the left of -10).

Let us denote $R_i = \{x_0, \ldots, x_{2i(i+1)}\} = \{-2i, \ldots, 2i-1\}$. Then we have that $x_{2i(i+1)} = 0$, and since $0 \in R_i$ we have that $x_{2i(i+1)+1} = 0 + f(0) = -2$. Since $-2 \in R_i$ we get $x_{2i(i+1)+2} = -2 - 2 = -4$, and so on. We obtain that $x_{2i(i+1)+i} = -2i \in R_i$ which implies that $x_{2i(i+1)+i+1} = -2i - 2 \notin R_i$. Therefore $x_{2i(i+1)+i+2} = -2i-2+g(-2i-2) = -2i-2+1 = -2i-1 \notin R_i$. Hence $x_{2i(i+1)+i+3} = -2i-1+g(-2i-1) = -2i+1 \in R_i$. This implies that $x_{2i(i+1)+i+4} = -2i + 3 \in R_i$. Continuing this way we obtain that $x_{2i(i+1)+i+2i+2} = 2i - 1 \in R_i$ and $x_{2i(i+1)+i+2i+3} = 2i + 1 \notin R_i$. Therefore $x_{2i(i+1)+3i+4} = 2i + 1 + g(2i + 1) = 2i \notin R_i$ and $x_{2i(i+1)+3i+5} = 2i + f(2i) = 2i - 2 \in R_i$.

We now have $x_{2i(i+1)+3i+6} = 2i - 4 \in R_i$ and continuing this way we obtain $x_{2i(i+1)+3i+i+4} = 0$. This implies that $x_{2i(i+1)+4(i+1)} = 0$ which is the same as $x_{2(i+1)(i+2)} = 0$. In addition,

$$\{x_0, \ldots, x_{2(i+1)(i+2)}\} = R_i \cup \{-2(i+1), -2i-1, 2i, 2i+1\} = \{-2i-2, \ldots, 2i, 2i+1\}$$

thus the proof of (2) is complete.

Placing $i = k-1$ in the first equation in (2) we obtain $x_{2k(k-1)} = 0$, and similarly as in the previous proof we get that $x_{2i(i+1)+3i+4} = 2i = 2k-2$. However, since $g(2k-2) = 1$ we get that $x_{2i(i+1)+3i+5} = 2k-1$ and subsequently that $x_{2i(i+1)+3i+6} = 2k - 1 + g(2k-1) = 2k-2$. This implies that $x_{2i(i+1)+3i+7} = 2k-2+f(2k-2) = 2k-1$ and $x_{2i(i+1)+3i+8} = 2k-1+f(2k-1) = 2k-2$. From now on the sequence is periodic and none of the terms will be zero.

This proves that there are exactly k terms equal to 0, and since it is stuck in a loop, no vertices outside $[-2k, 2k-1]$ are visited. □

3. LAWS OF LARGE NUMBERS

In this section we assume that the walk is in $\mathbb{R}^d$ for any $d \in \mathbb{N}$. We prove that the walk is almost surely bounded. As a consequence, the law of large numbers holds with the limiting velocity equal to 0. Moreover, all of the moments of the process X_n have growth that is slower than any function $f(n)$ that satisfies $\lim_{n\to\infty} f(n) = +\infty$. This means that the central limit theorem also does not have the classical form for this model.

The following lemma will be essential for the proofs of the boundedness of the walk. This lemma establishes the exponential decay of the probabilities that the walk reaches the *annulus* $\mathcal{A}_k$ defined in the following way:

$$\mathcal{A}_k = [-(k+1)L, (k+1)L]^d \setminus [-kL, kL]^d .$$

This way, $\mathcal{A}_0$ is the hypercube $[-L, L]^d$, while for $k \geq 1$, $\mathcal{A}_k$ is an annulus.

For any set $\mathcal{A} \subseteq \mathbb{R}^d$ let us define

$$T_{\mathcal{A}} = \inf\{n : X_n \in \mathcal{A}\} .$$

Lemma 2. *There exists a positive real number $c \in (0,1)$ and an integer k_0 such that*

$$\mathbb{P}\left(T_{\mathcal{A}_k} < +\infty\right) \leq c^k$$

holds for all $k \geq k_0$.

Proof. For each $x \in \mathbb{Z}^d$, let $x^+ \neq x$ be an arbitrarily chosen vertex from the set

$$\{y \in \mathbb{Z}^d : \|y\|_\infty \geq \|x\|_\infty \text{ and } \|x - y\|_\infty \leq L\},$$

where $\|x\|_\infty$ denotes the largest coordinate of x in absolute value. Observe that if x is outside the hypercube $[-kL, kL]^d$, then so is x^+, and that x^+ can be reached by the walker in one step from x. Denote by $G(x)$ the event that all cookies at x point to x^+, and all cookies at x^+ point to x. That is,

$$G(x) = \left\{\omega(j,x) = x^+ - x \text{ and } \omega(j,x^+) = x - x^+ \text{ for all } 0 \leq j \leq M-1\right\}.$$

On the event $G(x)$, the walk would get stuck in a loop between x and x^+ if it ever reached the site x.

We obviously have the following relation,

$$\mathbb{P}\left(T_{\mathcal{A}_{k+1}} < +\infty\right) \leq \mathbb{P}\left(T_{\mathcal{A}_k} < +\infty, G\left(X_{T_{\mathcal{A}_k}}\right)^C\right).$$

The Lemma will be established once we prove that for every $k \geq 0$ the following inequality holds:

$$\mathbb{P}\left(T_{\mathcal{A}_k} < +\infty, G\left(X_{T_{\mathcal{A}_k}}\right)^C\right) \leq \left(1 - \mu_{\min}^{2M}\right) \cdot \mathbb{P}\left(T_{\mathcal{A}_k} < +\infty\right). \tag{3}$$

For each $x \in \mathcal{A}_k$ let us introduce the event

$$\Omega_x = \left\{T_{\mathcal{A}_k} < +\infty \text{ and } X_{T_{\mathcal{A}_k}} = x\right\}.$$

The event Ω_x is in the sigma field generated by the cookies inside the set $\mathcal{A}_0 \cup \cdots \cup \mathcal{A}_{k-1}$. Therefore, Ω_x and $G(x)$ are independent.

We now have

$$\begin{aligned}
\mathbb{P}\left(T_{\mathcal{A}_k} < +\infty, G\left(X_{T_{\mathcal{A}_k}}\right)^C\right) &= \sum_{x \in \mathcal{A}_k} \mathbb{P}\left(\Omega_x \cap G(x)^C\right) \\
&= \sum_{x \in \mathcal{A}_k} \mathbb{P}\left(\Omega_x\right) \cdot \mathbb{P}\left(G(x)^C\right) \\
&\leq \left(1 - \mu_{\min}^{2M}\right) \cdot \sum_{x \in \mathcal{A}_k} \mathbb{P}\left(\Omega_x\right) \\
&= \left(1 - \mu_{\min}^{2M}\right) \cdot \mathbb{P}\left(T_{\mathcal{A}_k} < +\infty\right).
\end{aligned}$$

This completes the proof of (3), and hence the proof of the required inequality. □

A consequence of Lemma 2 is that the sequence X_n is almost surely bounded. We present this result in the following lemma.

Lemma 3. *Denote by B the event that X_n is a bounded sequence. More precisely, $B = \{\exists M_0$ such that $\|X_n\|_\infty \leq M_0$ for all $n \geq 0\}$, where $\|x\|_\infty$ denotes the biggest coordinate of the d-dimensional vector x in absolute value. Then $\mathbb{P}(B) = 1$.*

Proof. On the event B^C we must have $\{T_{\mathcal{A}_k} < +\infty\}$ for all $k \in \mathbb{N}$. However, Lemma 2 implies that $\mathbb{P}\left(T_{\mathcal{A}_k} < +\infty\right) < c^k$ for each $k \geq k_0$, hence

$$\mathbb{P}\left(B^C\right) = \mathbb{P}\left(\bigcap_{k \geq 1} \{T_{\mathcal{A}_k} < +\infty\}\right) \leq c^k,$$

for every $k \geq k_0$ which is only possible if $\mathbb{P}\left(B^C\right) = 0$. □

Corollary 1. *For every function $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n) = +\infty$ the following limit holds almost surely:*

$$\lim_{n\to\infty} \frac{\|X_n\|_\infty}{f(n)} = 0.$$

4. Large deviations

In this section we prove Theorem 1, and henceforth take $d = 1$. For $\lambda \in [0, L]$ we want to show the existence of the limit $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \geq \lambda n)$. As stated earlier, we will prove this under the assumption that there are at least 3 cookies on each site, i.e. $M \geq 3$. Before we can prove the theorem we need to introduce the following notation. For $k \in \mathbb{N}$ and $x \in \mathbb{Z}$ let us denote by V_x^k the time of the kth visit to the site x. The hitting time V_x^k can be inductively defined as:

$$\begin{aligned} V_x^1(\omega) &= \inf\{m : X_m(\omega) = x\}, \\ V_x^{i+1}(\omega) &= \inf\{m > V_x^i(\omega) : X_m(\omega) = x\} \text{ for } i \geq 1. \end{aligned}$$

Instead of V_x^1 we will often write V_x. As introduced earlier, for any set $\mathcal{A} \subseteq \mathbb{R}$ we will denote its hitting time by $T_{\mathcal{A}} = \inf\{n : X_n \in \mathcal{A}\}$. If $x > 0$ we will write T_x instead of $T_{[x,+\infty)}$. The following two inequalities are easy to establish:

$$\mathbb{P}(X_n \geq \lambda n) \leq \mathbb{P}(T_{\lambda n} \leq n) \quad \text{and} \tag{4}$$

$$\begin{aligned} \mathbb{P}(X_n \geq \lambda n) &\geq \mathbb{P}(T_{\lambda n} \leq n, \omega(j,x) = +1 \text{ for all } j \text{ and } x \in [\lambda n, \lambda n + L]) \\ &\geq C\mathbb{P}(T_{\lambda n} \leq n), \end{aligned} \tag{5}$$

for some constant C independent of n. Therefore, it is sufficient to prove that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(T_{\lambda n} \leq n)$ exists.

Let

$$A_n := \left\{ T_{\lambda n} \leq n, \inf_{k \leq T_{\lambda n}} X_k \geq 0 \right\}$$

denote the event that the walk reaches λn by time n before backtracking to the left of 0. It is trivially true that $A_n \subset \{T_{\lambda n} \leq n\}$ so $\mathbb{P}(A_n) \leq \mathbb{P}(T_{\lambda n} \leq n)$.

4.1. **Definitions.** If $a = (a_\ell)_{\ell=1}^K \in \mathbb{Z}^K$ where $K \in \mathbb{N} \cup \{\infty\}$ and $B \subset \mathbb{Z}$, then the restriction of a to B is denoted $a\big|_B$, and is the sequence of terms in a that belong to B with their order intact. For $t_1 \leq t_2$, let

$$X_{[t_1,t_2]}(\omega) = (X_{t_1}(\omega), X_{t_1+1}(\omega), \ldots, X_{t_2}(\omega))$$

denote the sequence of locations of the walker from steps t_1 through t_2.

Definition 1. *For $\omega, \omega' \in \{T_\ell < \infty\}$ and $0 \leq m < \ell$, let $\omega' \prec_{\ell,m} \omega$ denote the following relationship between environments ω and ω'.*

1. *$\omega'(j,x) = \omega(j,x)$ for all $x > m$ and all $j \geq 0$;*
2. *$X_{[0,T_\ell(\omega')]}(\omega')\big|_{[m,\ell]} = X_{[0,T_\ell(\omega)]}(\omega)\big|_{[m,\ell]}$;*
3. *The sequence $X_{[0,T_\ell(\omega')]}(\omega')$ is a subsequence of $X_{[0,T_\ell(\omega)]}(\omega)$.*

In other words, we will write $\omega' \prec_{\ell,m} \omega$ if (1) the two environments are identical to the right of m, (2) the walkers on both environments visit the same sites in the same order to the right of m and until exceeding ℓ, but (3) the walker on ω' may avoid some parts of the path followed by the walker on ω to the left of m. Observe that $\prec_{\ell,m}$ gives a partial ordering of the environments in $\{T_\ell < \infty\}$.

4.2. **Monotonicity results.** The next theorem provides the asymptotic equivalence of probabilities $\mathbb{P}(T_{\lambda n} \leq n)$ and $\mathbb{P}(A_n)$ on the logarithmic scale.

Theorem 3. *There exists $C \in \mathbb{R}_+$, depending on L, M and μ, such that the following inequality holds for all n:*

$$C^{\sqrt{n}}\mathbb{P}(A_n) \geq \mathbb{P}(T_{\lambda n} \leq n). \tag{6}$$

Proof. We will use the following result whose proof will be presented later.

Lemma 4. *Assume that $n > \left(\frac{2L}{\lambda}\right)^2$. For each $\omega \in \{T_{\lambda n} \leq n\}$ there exists $\omega' \in \{T_{\lambda n} \leq n\}$ such that*

$$\omega' \prec_{\lambda n, 2L\sqrt{n}} \omega \quad \text{and} \quad X_{[0, T_{\lambda n}(\omega')]}(\omega') \cap (-\infty, -1] = \emptyset.$$

For given $\omega \in \{T_{\lambda n} \leq n\}$ we can apply Lemma 4 to obtain a new environment $\hat{\omega} \in \{T_{\lambda n} \leq n\}$ such that

$$\inf_{0 \leq k \leq T_{\lambda n}(\hat{\omega})} X_k(\hat{\omega}) = 0.$$

Let us denote by $\tilde{\omega}$ the environment defined by:

(i) For $x \notin [0, 2L\sqrt{n}]$ and $j \in \{0, \dots, M-1\}$: $\tilde{\omega}(j,x) = \omega(j,x)$.
(ii) For $x \in [0, 2L\sqrt{n}]$ and $j \in \{0, \dots, M-1\}$: $\tilde{\omega}(j,x) = \hat{\omega}(j,x)$.

Since $\hat{\omega}$ and $\tilde{\omega}$ coincide on sites in $[0, +\infty)$ and $X(\hat{\omega})$ does not visit negative sites, we conclude that $X(\tilde{\omega})$ does not visit negative sites. Therefore, for each $\omega \in \{T_{\lambda n} \leq n\}$ there exists $\tilde{\omega} \in A_n$ such that ω and $\tilde{\omega}$ coincide on all sites except possibly for the sites in $[0, 2L\sqrt{n}]$.

We can now define a function $f : \{T_{\lambda n} \leq n\} \to A_n$ in the following way. For each $\omega \in \{T_{\lambda n} \leq n\}$ we pick one $\tilde{\omega}$ with the properties established in the previous paragraph and define $f(\omega) = \tilde{\omega}$.

Let us fix n. We can now define $\mathbb{P}_n$ on the restriction Ω_n of Ω that corresponds to the portion of the integer axis between the numbers $-Ln$ and Ln. The purpose of this restriction is so that $\mathbb{P}_n(\omega) > 0$ for each $\omega \in \Omega_n$. Formally,

$$\Omega_n = [-L, L]^{[0,M-1]\times[-Ln,Ln]},$$

and $\mathbb{P}_n$ is defined to be the restriction of $\mathbb{P}$. Then we have $\mathbb{P}_n(T_{\lambda n} \leq n) = \mathbb{P}(T_{\lambda n} \leq n)$ and $\mathbb{P}_n(A_n) = \mathbb{P}(A_n)$, where each $\omega \in \Omega$ is identified with an element of Ω_n by truncation, which will also be denoted ω. It suffices to prove that there is $C \in \mathbb{R}_+$ (independent of n) such that

$$\mathbb{P}_n(T_{\lambda n} \leq n) \leq C^{\sqrt{n}}\mathbb{P}_n(A_n). \tag{7}$$

Observe that if environment $\omega' \in \Omega_n$ differs from environment $\omega \in \Omega_n$ at exactly one site, $z \in [-Ln, Ln]$, then $\mathbb{P}_n(\omega) \leq \mathbb{P}_n(\omega')/(\mu_{\min})^M$. Let $C_1 = \left(\frac{1}{\mu_{\min}}\right)^M$ and $C_2 = (2L+1)^M$. We will prove inequality (7) for $C = (C_1C_2)^{2L}$.

$$\begin{aligned}
\mathbb{P}_n(T_{\lambda n} \leq n) &= \sum_{\omega \in \{T_{\lambda n} \leq n\}} \mathbb{P}_n(\omega) \leq \sum_{\omega \in \{T_{\lambda n} \leq n\}} C_1^{2L\sqrt{n}}\mathbb{P}_n(f(\omega)) \\
&= C_1^{2L\sqrt{n}} \sum_{\omega \in \{T_{\lambda n} \leq n\}} \sum_{\omega' \in A_n} \mathbb{P}_n(\omega') \cdot 1_{f(\omega)=\omega'} \\
&= C_1^{2L\sqrt{n}} \sum_{\omega' \in A_n} \sum_{\omega \in \{T_{\lambda n} \leq n\}} \mathbb{P}_n(\omega') \cdot 1_{f(\omega)=\omega'} \\
&= C_1^{2L\sqrt{n}} \sum_{\omega' \in A_n} \mathbb{P}_n(\omega') \cdot \sum_{\omega \in \{T_{\lambda n} \leq n\}} 1_{f(\omega)=\omega'} \\
&= C_1^{2L\sqrt{n}} \sum_{\omega' \in A_n} \mathbb{P}_n(\omega') \cdot \left|\left\{f^{-1}(\omega')\right\}\right|.
\end{aligned}$$

If $f(\omega)=\omega'$ then the environments ω and ω' coincide outside of $[0,2L\sqrt{n}]$. Since there could be at most $C_2^{2L\sqrt{n}}$ different environments that coincide with ω outside of $[0,2L\sqrt{n}]$, we obtain

$$\mathbb{P}_n\left(T_{\lambda n}\leq n\right)\leq C_1^{2L\sqrt{n}}\cdot C_2^{2L\sqrt{n}}\sum_{\omega'\in A_n}\mathbb{P}_n(\omega')=C^{\sqrt{n}}\mathbb{P}_n(A_n).$$

This completes the proof of inequality (7) which implies (6). □

In order to prove Lemma 4 we first need to establish the following result.

Lemma 5. *Fix $\omega\in\Omega$. Suppose $a,b\in\mathbb{Z}$ with $|a-b|\leq L$, and $0\leq t_a<t_b$ are such that $X_{t_a}(\omega)=a$, $X_{t_b}(\omega)=b$, and one of the following two conditions is satisfied:*

(a) $L_{t_a}(\omega,a)<M-1$;
(b) $X_{t_a+1}(\omega)=b$.

Then there exists $\omega'\in\Omega$ such that

(i) $X_{[0,t_a]}(\omega')=X_{[0,t_a]}(\omega)$;
(ii) $X_{[t_a+1,\infty)}(\omega')=X_{[t_b,\infty)}(\omega)$;
(iii) $\omega'(j,x)=\omega(j,x)$ *for all* $x\notin X_{[t_a,t_b]}(\omega)$ *and all* $j\geq 0$.

Furthermore, if ω contains no self-loop cookies ($\omega(j,x)=0$) and $a\neq b$, then ω' also contains no self-loop cookies.

Proof. Let $C_x=L_{t_b}(\omega,x)-L_{t_a}(\omega,x)$ be the number of times the site x is visited by the sequence $X_{[t_a,t_b-1]}(\omega)$. Under the assumption (a) we obtain the environment ω' from ω by removing the cookies visited by the walker $X(\omega)$ in the time interval $[t_a+1,t_b]$ and rewiring the top cookie at a at time t_a to point at b. That is,

$$\omega'(j,x)=\begin{cases}\omega(j,x) & \text{for } x\notin X_{[t_a,t_b]}(\omega)\text{ and all } j\geq 0\\ \omega(j,x) & \text{for } x\in X_{[t_a,t_b]}(\omega),\ 0\leq j<L_{t_a}(\omega,x)\\ \omega(j+C_x,x) & \text{for } x\in X_{[t_a,t_b]}(\omega),\ x\neq a,\ j\geq L_{t_a}(\omega,x)\\ \omega(j+C_a,a) & \text{for } x=a,\ j>L_{t_a}\\ b-a & \text{for } x=a,\ j=L_{t_a}(\omega,a).\end{cases}$$

From the definition of ω', it is clear that (iii) is satisfied, and (i) is satisfied because, from the perspective of the walker, ω' and ω are identical up until time t_a. Finally, (ii) is satisfied because the remaining environments at time t_b in $X(\omega)$ and at time t_a+1 in $X(\omega')$ are identical. The assumption (a) guarantees that rewiring the top cookie at a is allowed. (Without this assumption we would be rewiring the cookie labeled $(M-1)$, which would modify all cookies $j\geq M-1$, thus affecting the future path of the walk.)

If (b) is assumed instead of (a), then no rewiring is necessary since the cookie at a at the time t_a points to b in both ω and ω', i.e. $\omega\left(L_{t_a}+C_a,a\right)=b-a$ and we can keep the same definition for ω' as when working under the assumption (a). □

Proof of Lemma 4. Fix $\omega\in\{T_{\lambda n}\leq n\}$, and let

$$\mathcal{G}(\omega)=\left\{\omega'\in\{T_{\lambda n}\leq n\}:\omega'\prec_{\lambda n,2L\sqrt{n}}\omega\right\}.$$

Observe that $\omega\in\mathcal{G}(\omega)$, so $\mathcal{G}(\omega)\neq\emptyset$. Choose $\sigma\in\mathcal{G}(\omega)$ such that

$$T_{\lambda n}(\sigma)=\min_{\omega'\in\mathcal{G}(\omega)}T_{\lambda n}(\omega'). \tag{8}$$

Such a σ exists because $T_{\lambda n}(\omega')$ is an integer for each ω'. If $X_{[0,T_{\lambda n}(\sigma)]}(\sigma)\cap(-\infty,-1]=\emptyset$ we may take $\omega'=\sigma$. Assume therefore that there exists $x<0$ such that $x\in X_{[0,T_{\lambda n}(\sigma)]}(\sigma)$. Let us define

$$\alpha_1=\max\left\{0\leq k<V_x(\sigma):X_k(\sigma)>x\right\},\qquad a_1=X_{\alpha_1}(\sigma),$$
$$\beta_1=\min\left\{k>V_x(\sigma):X_k(\sigma)>x,X_k(\sigma)\neq a_1\right\},\qquad b_1=X_{\beta_1}(\sigma).$$

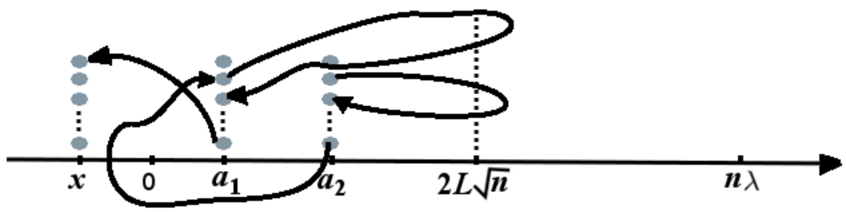

FIGURE 3. The walker must travel a long distance (at least $L\sqrt{n}$) between the second and third visits to a site marked $a_i \leq L\sqrt{n}$. Each of these excursions takes at least $\sqrt{n}$ steps.

The times α_1 and β_1 are well defined because $x < 0$ and $\sigma \in \{T_{\lambda n} \leq n\}$. The constraint $X_k(\sigma) \neq a_1$ in the definition of β_1 is to guarantee $b_1 \neq a_1$, so we avoid using a self-loop when rewiring via Lemma 5. Clearly, $a_1 \in (x, x+L], b_1 > x$ and $|a_1 - b_1| \leq L$. Assume that $L_{\alpha_1}(\sigma, a_1) < M - 1$. Then we can apply Lemma 5 to $a = a_1$, $b = b_1$, $t_a = \alpha_1$, and $t_b = \beta_1$. The application of the lemma allows us to obtain an environment $\sigma' \in \{T_{\lambda n} \leq n\}$ from the original environment σ such that $\sigma' \prec_{\lambda n, 2L\sqrt{n}} \sigma$. Since one visit to x is avoided in σ' we would have $T_{\lambda n}(\sigma') \leq T_{\lambda n}(\sigma) - 1$, which contradicts (8). Therefore we must have $L_{\alpha_1}(\sigma, a_1) \geq M - 1$ (recall that this means the walker is visiting a_1 for at least the M^{th} time at step α_1). We now consider the sequence of times $V_{a_1}^1(\sigma)$, $V_{a_1}^2(\sigma)$, ..., $V_{a_1}^M(\sigma)$ at which the visits to a_1 have occurred. Assume that for some $s \in \{1, \dots, M-1\}$ we have

$$X_{\left[V_{a_1}^s(\sigma), V_{a_1}^{s+1}(\sigma)\right]}(\sigma) \subseteq \left(-\infty, 2L\sqrt{n}\right].$$

By applying Lemma 5 (with assumption (b)) to $a = X_{V_{a_1}^s(\sigma)-1}$, $b = a_1$, $t_a = V_{a_1}^s(\sigma) - 1$, and $t_b = V_{a_1}^{s+1}$ we obtain an environment $\sigma' \in \{T_{\lambda n} < n\}$ such that $\sigma' \prec_{\lambda n, 2L\sqrt{n}} \sigma$ and $L_{T_{\lambda n}(\sigma')}(\sigma', a_1) \leq L_{T_{\lambda n}(\sigma)}(\sigma, a_1) - 1$, which contradicts (8).

Let us now define

$$\alpha_2 = \max\left\{0 \leq k < V_{a_1}^2(\sigma) : X_k(\sigma) > a_1\right\}, \qquad a_2 = X_{\alpha_2}(\sigma),$$
$$\beta_2 = \min\left\{k > V_{a_1}^2(\sigma) : X_k(\sigma) > a_1, X_k(\sigma) \neq a_2\right\}, \qquad b_2 = X_{\beta_2}(\sigma).$$

Note that we must have $a_1 \leq x+L < L\sqrt{n}$. We can be certain that α_2 is well defined because $X_{\left[V_{a_1}^1(\sigma), V_{a_1}^2(\sigma)\right]} \cap (2L\sqrt{n}, +\infty) \neq \emptyset$. The time β_2 is also well defined because $\sigma \in \{T_{\lambda n} \leq n\}$. According to the construction we must have $a_2 \in (a_1, a_1 + L]$, $b_2 > a_1$ and $|a_2 - b_2| \leq L$.

Using the same argument as above we have that $L_{\alpha_2}(\sigma, a_2) \geq M-1$, and that for each $s \in \{1, 2, \dots M-1\}$ we have

$$X_{\left[V_{a_2}^s(\sigma), V_{a_2}^{s+1}(\sigma)\right]}(\sigma) \cap \left(2L\sqrt{n}, +\infty\right) \neq \emptyset.$$

Having defined $a_1 < \cdots < a_i$ and assuming that $a_i < L\sqrt{n}$ we inductively define the times α_{i+1} and β_{i+1} in the following way:

$$\alpha_{i+1} = \max\left\{0 \leq k < V_{a_i}^2(\sigma) : X_k(\sigma) > a_i\right\}, \qquad a_{i+1} = X_{\alpha_{i+1}}(\sigma)$$
$$\beta_{i+1} = \min\left\{k > V_{a_i}^2(\sigma) : X_k(\sigma) > a_i, X_k(\sigma) \neq a_{i+1}\right\}, \qquad b_{i+1} = X_{\beta_{i+1}}(\sigma).$$

Clearly, $a_{i+1} \in (a_i, a_i + L], b_{i+1} > a_i$ and $|a_{i+1} - b_{i+1}| \leq L$. As above, we are certain that

$$L_{\alpha_{i+1}}(\sigma, a_{i+1}) \geq M - 1, \tag{9}$$

and for each $s \in \{1, 2, \dots, M-1\}$ the following property holds:

$$X_{\left[V_{a_{i+1}}^s(\sigma), V_{a_{i+1}}^{s+1}(\sigma)\right]}(\sigma) \cap \left(2L\sqrt{n}, +\infty\right) \neq \emptyset. \tag{10}$$

We can continue the induction until we have $x < a_1 < \cdots < a_I$ where I is the smallest index such that $a_I \geq L\sqrt{n}$. Since $a_{i+1} - a_i \leq L$ for each $i \leq I-1$, we must have $I \geq \sqrt{n}$. From (9), we have that for each $i \leq I-1$, before the second visit to the site a_i, the walk $X(\sigma)$ visits the site a_{i+1} at least M times. Furthermore, (10) implies that $V_{a_i}^{s+1}(\sigma) - V_{a_i}^{s}(\sigma) \geq \sqrt{n}$ for each $i \leq I$ and $s \leq M-1$, that is, the walk spends at least $\sqrt{n}$ steps between consecutive visits to each site a_i.

Now we will use our assumption that $M \geq 3$. We know that a_1 is visited at least three times before x is visited for the first time. Between the second and third visit to a_1 the walk spent at least $\sqrt{n}$ steps, as depicted in Figure 3. Therefore $V_x(\sigma) \geq V_{a_1}^3(\sigma) \geq V_{a_1}^2(\sigma) + \sqrt{n}$. The second visit to a_1 has occurred after the site a_2 is visited at least M times, hence the second visit to a_1 occurred after the third visit to a_2. Therefore $V_{a_1}^2(\sigma) \geq V_{a_2}^3(\sigma) \geq V_{a_2}^2(\sigma) + \sqrt{n}$. Thus $V_x(\sigma) \geq V_{a_2}^2(\sigma) + 2\sqrt{n}$. Since the second visit to a_2 occurred after M visits to a_3 we know that the second visit to a_2 occurred after the third visit to a_3. Thus $V_{a_2}^2(\sigma) \geq V_{a_3}^3(\sigma) \geq V_{a_3}^2(\sigma) + \sqrt{n}$ and $V_x(\sigma) \geq V_{a_3}^2(\sigma) + 3\sqrt{n}$. Continuing in this fashion, we obtain that $V_x(\sigma) \geq V_{a_I}^2(\sigma) + I\sqrt{n} \geq V_{a_I}^2(\sigma) + n \geq n$, which contradicts the assumption that $V_x(\sigma) < T_{\lambda n}(\sigma) \leq n$. This completes the proof of Lemma 4. □

4.3. **Large deviations.** In this subsection we provide the proof to Theorem 1.

Proof of Theorem 1. It suffices to prove that $\mathbb{P}(A_{m+n}) \geq \mathbb{P}(A_n) \cdot \mathbb{P}(A_m)$. We notice the following inclusion:

$$\begin{aligned} A_{n+m} &= \left\{T_{\lambda(n+m)} \leq n+m, \inf\{X_k : 0 \leq k \leq T_{\lambda(n+m)}\} \geq 0\right\} \\ &\supseteq A_n \cap \left\{T_{\lambda(n+m)} \leq n+m, \inf\{X_k : 0 \leq k \leq T_{\lambda(n+m)}\} \geq 0\right\}. \end{aligned}$$

Let us define the walk $\hat{X}(\omega)$ for $\omega \in A_n \cap \{T_{\lambda n} \leq n, \inf\{X_k : 0 \leq k \leq T_{\lambda n}\} \geq 0\}$ in the following way: $\hat{X}_k(\omega) = X_{k+T_{\lambda n}}(\omega) - X_{T_{\lambda n}}(\omega)$. The walk $\hat{X}$ starts at 0. In analogy to the stopping time T_x for the walk X we define $\hat{T}_x$ for the walk $\hat{X}$. The precise definition is:

$$\hat{T}_x(\omega) = T_{X_{T_{\lambda n}}+x}(\omega) - T_{\lambda n}(\omega).$$

In analogy to A_n we define the event $\hat{A}_m$ for the walk $\hat{X}$:

$$\hat{A}_m = \left\{\hat{T}_{\lambda m} \leq m, \inf\left\{\hat{X}_k : 0 \leq k \leq \hat{T}_{\lambda m}\right\} \geq 0\right\}.$$

On the event $A_n \cap \hat{A}_m$, by time $T_{\lambda n} + \hat{T}_{\lambda m}$ the walk X reaches the site $X_{T_{\lambda n}} + \hat{X}_{\hat{T}_{\lambda m}} \geq \lambda(n+m)$. Therefore $A_n \cap \hat{A}_m \subseteq A_{n+m}$. We will now prove that $\mathbb{P}\left(A_n \cap \hat{A}_m\right) = \mathbb{P}(A_n) \cdot \mathbb{P}\left(\hat{A}_m\right)$. For each $x \in [\lambda n, \lambda n + L]$, conditioned on $X_{T_{\lambda n}} = x$, the events A_n and $\hat{A}_m$ are independent. Therefore

$$\begin{aligned} \mathbb{P}\left(A_n \cap \hat{A}_m\right) &= \sum_{x \in [\lambda n, \lambda n + L]} \mathbb{P}\left(A_n \cap \hat{A}_m \middle| X_{T_{\lambda n}} = x\right) \cdot \mathbb{P}\left(X_{T_{\lambda n}} = x\right) \\ &= \sum_{x \in [\lambda n, \lambda n + L]} \mathbb{P}\left(A_n \middle| X_{T_{\lambda n}} = x\right) \cdot \mathbb{P}\left(\hat{A}_m \middle| X_{T_{\lambda n}} = x\right) \cdot \mathbb{P}\left(X_{T_{\lambda n}} = x\right). \end{aligned}$$

Since $\mathbb{P}\left(\hat{A}_m \,|X_{T_{\lambda n}} = x\right) = \mathbb{P}\left(\hat{A}_m\right)$ we obtain

$$\begin{aligned} \mathbb{P}\left(A_n \cap \hat{A}_m\right) &= \mathbb{P}\left(\hat{A}_m\right) \cdot \sum_{x \in [\lambda n, \lambda n + L]} \mathbb{P}\left(A_n \middle| X_{T_{\lambda n}} = x\right) \cdot \mathbb{P}\left(X_{T_{\lambda n}} = x\right) \\ &= \mathbb{P}\left(\hat{A}_m\right) \cdot \mathbb{P}(A_n), \end{aligned}$$

which implies the inequality

$$\mathbb{P}(A_{n+m}) \geq \mathbb{P}(A_n) \cdot \mathbb{P}(A_m)$$

for all $n, m > 0$. Fekete's subadditive lemma (see [18]) implies the existence of the limit

$$\phi(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(A_n\right).$$

The proof is completed using the inequalities (4) and (5) and Theorem 3. □

5. Case $L = 2$ or $M = 1$

In the case when $L = 2$ or the number of cookies per site is 0 we can obtain the exponential decay of probabilities $\mathbb{P}\left(X_n \geq \lambda\xi(n)\right)$ for every positive function ξ that satisfies $\xi(n)+\xi(m) \geq \xi(n+m)$. In particular this holds for $\xi(x) = x^\theta$ for $\theta \in (0, 1]$.

Theorem 4. *Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive super-additive function and assume that either $L = 2$ or $M = 1$. Then there is a function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that for every $\lambda > 0$ the following holds:*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \geq \lambda\xi(n)\right) = \varphi(\lambda). \tag{11}$$

Proof. We will prove the theorem for the case $L = 2$. The proof when $M = 1$ is a simple generalization of the proof from the case of deterministic walks in random environments in [11]. First of all, the following inequalities are obtained in the same way as in the proof of Theorem 1:

$$\liminf \frac{1}{n} \log \mathbb{P}\left(T_{\lambda\xi(n)} \leq n\right) \leq \liminf \frac{1}{n} \log \mathbb{P}\left(X_n \geq \lambda\xi(n)\right) \tag{12}$$

$$\limsup \frac{1}{n} \log \mathbb{P}\left(X_n \geq \lambda\xi(n)\right) \leq \limsup \frac{1}{n} \log \mathbb{P}\left(T_{\lambda\xi(n)} \leq n\right). \tag{13}$$

Let $A_n := \{T_{\lambda\xi(n)} \leq n, \inf_{k \leq T_{\lambda\xi(n)}} X_k \geq 0\}$. We have $\mathbb{P}\left(A_n\right) \leq \mathbb{P}\left(T_{\lambda\xi(n)} \leq n\right)$. We will now prove that $\mathbb{P}\left(T_{\lambda\xi(n)} \leq n\right) \leq C\mathbb{P}\left(A_n\right)$ for constant C independent of n.

Lemma 6. *Suppose $L \leq 2$. For each $\omega \in \{T_{\lambda\xi(n)} \leq n\}$ there exists an $\omega' \in A_n$ such that*

$$\omega' \prec_{\lambda\xi(n),2} \omega.$$

Proof. If $L = 1$, given $\omega \in \{T_{\lambda\xi(n)} \leq n\}$ we can define ω' such that $\omega'(j, 0) = +1$ for every $j \geq 0$, and $\omega'(j, x) = \omega(j, x)$ for every $j \geq 0$ and $x \neq 0$. It is easy to check that $\omega' \in A_n$ and $\omega' \prec_{\lambda\xi(n),2} \omega$ (in fact, ω' agrees with ω everywhere except at 0).

Suppose now that $L = 2$, and assume that there is an element $\omega \in \{T_{\lambda\xi(n)} \leq n\}$ for which the desired $\omega' \in A_n$ does not exist. Consider the set

$$\mathcal{G}(\omega) = \left\{\omega' \in \{T_{\lambda\xi(n)} \leq n\} : \omega' \prec_{\lambda\xi(n),2} \omega\right\}$$

and an element $\sigma \in \mathcal{G}(\omega)$ such that

$$T_{\lambda\xi(n)}(\sigma) = \min_{\omega' \in \mathcal{G}(\omega)} T_{\lambda\xi(n)}(\omega').$$

Let us define the following times:

$$\begin{aligned} \alpha &= \sup\{k < T_{(-\infty,0)}(\sigma) : X_k(\sigma) \geq 0\}, \\ \beta &= \inf\{k > \alpha : X_k(\sigma) \geq 0, X_k(\sigma) \neq X_\alpha(\sigma)\}. \end{aligned}$$

Our assumption implies that $\sigma \notin A_n$ hence $\alpha < +\infty$. Clearly, $X_\alpha \in \{0, 1\}$ and $|X_\beta - X_\alpha| \leq 2$ since $L = 2$. If $L_\alpha(\sigma, X_\alpha) < M - 1$, then we can apply Lemma 5 to $a = X_\alpha$, $b = X_\beta$, $t_a = \alpha$, and $t_b = \beta$. We obtain an environment $\sigma' \prec_{\lambda\xi(n),2} \sigma$ in which at least one visit to $(-\infty, 0)$ is avoided implying that $T_{\lambda\xi(n)}(\sigma') < T_{\lambda\xi(n)}(\sigma)$. Therefore $L_\alpha(\sigma, X_\alpha) \geq M-1$, which implies that $\{X_\alpha, X_\beta\} = \{0, 1\}$, since otherwise the walk could never move to the right of X_α after time α. Let σ' be the environment obtained from σ in the following way:

$$\sigma'(j,x)=\begin{cases}\sigma(j,x) & \text{for } (j,x)\neq(M-1,X_\alpha)\\ X_\beta-X_\alpha & \text{for } (j,x)=(M-1,X_\alpha).\end{cases}$$

Since every visit to $(-\infty,0)$ in σ after time α must start from X_α and end at X_β we conclude that $\sigma'\prec_{\lambda\xi(n),2}\sigma$ and $X_{[0,T_{\lambda\xi(n)}(\sigma')]}\cap(-\infty,0)=\emptyset$, which contradicts our minimality assumption on σ. This completes the proof of Lemma 6. □

In the same way as in the proof of inequality (6) we now establish

$$\mathbb{P}\left(T_{\lambda\xi(n)}\leq n\right)\leq C\mathbb{P}\left(A_n\right).$$

An argument analogous to the one presented in the proof of Theorem 1 allows us to prove the existence of the function φ such that

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(A_n\right)=\varphi(\lambda).$$

The inequalities (12) and (13) allow us to conclude (11). □

6. Properties of the rate function

The next theorem states that the rate function ϕ from (1) is concave in λ.

Theorem 5. *Assume that $\alpha,\beta>0$ are real numbers such that $\alpha+\beta=1$. Then for any λ, $\gamma>0$ the following inequality holds*

$$\phi(\alpha\lambda+\beta\gamma)\geq\alpha\phi(\lambda)+\beta\phi(\gamma). \tag{14}$$

Moreover, $\phi(0)=0$, $\phi(\lambda)<0$ for $\lambda>0$, and $\phi(\lambda)=-\infty$ for $\lambda>L$.

Proof. The equality $\phi(\lambda)=-\infty$ for $\lambda>L$ is trivial because $\mathbb{P}\left(X_n>Ln\right)=0$. We will now prove the equality $\phi(0)=0$. The event $\{X_n\geq 0\}$ contains the event $\{\omega(j,0)=1=-\omega(j,1)\text{ for all } j\in\{0,1,2,\dots,M-1\}\}$, that is, the event that all cookies at 0 point to 1 and all cookies at 1 point back to 0. The probability of this event is at least $\mu_{\min}^{2M}$, hence $\mathbb{P}\left(X_n\geq 0\right)\geq\mu_{\min}^{2M}$. On the other hand, the complement of $\{X_n\geq 0\}$ contains the event that the first cookie at 0 points to -1, all cookies at -1 point to -2 and all cookies at -2 point back to -1. Thus, $\mathbb{P}\left(X_n\geq 0\right)\leq 1-\mu_{\min}^{2M+1}$, and we conclude that

$$0=\lim_{n\to\infty}\frac{1}{n}\log\mu_{\min}^{2M}\leq\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(X_n\geq 0\right)\leq\lim_{n\to\infty}\frac{1}{n}\log\left(1-\mu_{\min}^{2M+1}\right)=0.$$

For the remainder of the proof, we assume that $\lambda\in(0,L]$. Let $k=\left\lfloor\frac{\lambda n}{L}\right\rfloor$. We will prove that $\phi(\lambda)\in(-\infty,0)$ using Lemma 2. Let $\mathcal{A}_k=[-(k+1)L,(k+1)L]\setminus[-kL,kL]$. Since $\{X_n\geq\lambda n\}\subseteq\{T_{\mathcal{A}_k}<+\infty\}$ we use Lemma 2 to conclude that $\mathbb{P}\left(X_n\geq\lambda n\right)\leq c^k$ for some constant $c\in(0,1)$ and all sufficiently large n. For sufficiently large n we have that $\left\lfloor\frac{\lambda n}{L}\right\rfloor\geq\frac{n\lambda}{2L}$ hence $\mathbb{P}\left(X_n\geq\lambda n\right)\leq\left(c^{\frac{\lambda}{2L}}\right)^n$. This implies that

$$\phi(\lambda)\leq\lim_{n\to\infty}\frac{1}{n}\log\left(c^{\frac{\lambda}{2L}}\right)^n<0.$$

The finiteness of $\phi(\lambda)$ follows from the fact that $\{T_{\lambda n}\leq n\}$ contains the event

$$G=\{\omega(0,0)=\omega(0,L)=\omega(0,2L)=\cdots=\omega(0,nL)=L\},$$

which is the event that the top cookies at each of the sites 0, L, $2L$, …, nL point to the location that is L units to its right. The probability of the last event is at least $\mu_{\min}^n$ hence $\phi(\lambda)\geq\log\mu_{\min}$.

We will now demonstrate the concavity of ϕ. Assume that $\lambda,\gamma\in(0,L]$ (otherwise concavity is trivial, as the right hand side of (14) is $-\infty$). Assume that $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ are sequences of rational numbers for which $n\alpha_n$, $n\beta_n\in\mathbb{N}$, $0\leq\alpha-\alpha_n\leq\frac{1}{n}$, and $0\leq\beta-\beta_n\leq\frac{1}{n}$. Notice that

$$\begin{aligned}&\left\{T_{(\alpha\lambda+\beta\gamma)n}\leq n,\inf_{0\leq k\leq T_{(\alpha\lambda+\beta\gamma)n}}X_k\geq 0\right\}\\ \supseteq\ &\left\{T_{(\alpha\lambda+\beta\gamma)n}\leq n,\inf_{0\leq k\leq T_{(\alpha\lambda+\beta\gamma)n}}X_k\geq 0,T_{\alpha_n\lambda n}\leq\alpha_n n\right\}.\end{aligned}$$

Analogously as in the proof of Theorem 1 we define the process $\hat{X}$ as

$$\hat{X}_k(\omega)=X_{k+T_{\alpha_n\lambda n}}(\omega)-X_{T_{\alpha_n\lambda n}}(\omega).$$

Also, denote by $\hat{T}_x$ the hitting time of the walk $\hat{X}$, i.e. $\hat{T}_x=T_{X_{T_{\alpha_n\lambda n}}+x}-T_{\alpha_n\lambda n}$. We now obtain

$$\begin{aligned}&\left\{T_{(\alpha\lambda+\beta\gamma)n}\leq n,\inf_{0\leq k\leq T_{(\alpha\lambda+\beta\gamma)n}}X_k\geq 0,T_{\alpha_n\lambda n}\leq\alpha_n n\right\}\\ \supseteq\ &\left\{T_{(\alpha\lambda+\beta\gamma)n}\leq n,\inf_{0\leq k\leq T_{(\alpha\lambda+\beta\gamma)n}}X_k\geq 0,T_{\alpha_n\lambda n}\leq\alpha_n n,\right.\\ &\left.\inf_{0\leq k\leq T_{\alpha_n\lambda n}}X_k\geq 0,\hat{T}_{\beta_n\gamma n}\leq\beta_n n,\inf_{0\leq k\leq\hat{T}_{\beta_n\gamma n}}\hat{X}_k\geq 0\right\}.\end{aligned}\tag{15}$$

Let us denote by $\tilde{X}$ the walk defined as $\tilde{X}_k=\hat{X}_{k+\hat{T}_{\beta_n\gamma n}}-\hat{X}_{\hat{T}_{\beta_n\gamma n}}$, and by $\tilde{T}_x$ the stopping time $\tilde{T}_x=\hat{T}_{\hat{X}_{\hat{T}_{\beta_n\gamma n}}+x}-\hat{T}_{\beta_n\gamma n}$. Now we can conclude from the inclusion (15) that

$$\begin{aligned}&\left\{T_{(\alpha\lambda+\beta\gamma)n}\leq n,\inf_{0\leq k\leq T_{(\alpha\lambda+\beta\gamma)n}}X_k\geq 0,T_{\alpha_n\lambda n}\leq\alpha_n n\right\}\\ \supseteq\ &\left\{T_{(\alpha\lambda+\beta\gamma)n}\leq n,\inf_{0\leq k\leq T_{(\alpha\lambda+\beta\gamma)n}}X_k\geq 0,T_{\alpha_n\lambda n}\leq\alpha_n n,\right.\\ &\inf_{0\leq k\leq T_{\alpha_n\lambda n}}X_k\geq 0,\hat{T}_{\beta_n\gamma n}\leq\beta_n n,\inf_{0\leq k\leq\hat{T}_{\beta_n\gamma n}}\hat{X}_k\geq 0,\\ &\left.\tilde{T}_{(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)n}\leq(1-\alpha_n-\beta_n)n,\inf_{0\leq k\leq\tilde{T}_{(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)n}}\tilde{X}_k\geq 0\right\}\\ =\ &\left\{T_{\alpha_n\lambda n}\leq\alpha_n n,\inf_{0\leq k\leq T_{\alpha_n\lambda n}}X_k\geq 0,\hat{T}_{\beta_n\gamma n}\leq\beta_n n,\inf_{0\leq k\leq\hat{T}_{\beta_n\gamma n}}\hat{X}_k\geq 0,\right.\\ &\left.\tilde{T}_{(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)n}\leq(1-\alpha_n-\beta_n)n,\inf_{0\leq k\leq\tilde{T}_{(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)n}}\tilde{X}_k\geq 0\right\}.\end{aligned}$$

From our choice of sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ we derive the following two inequalities

$$\begin{aligned}(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)\,n&\leq L\,(\alpha+\beta-\alpha_n-\beta_n)\,n,\ \text{ and}\\ (1-\alpha_n-\beta_n)n&\leq 2.\end{aligned}$$

Observe that $(1-\alpha_n-\beta_n)\,n\in\{0,1,2\}$. In each of the three cases we have

$$\mathbb{P}\left(\tilde{T}_{(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)n}\leq(1-\alpha_n-\beta_n)n,\inf_{0\leq k\leq\tilde{T}_{(\alpha\lambda+\beta\gamma-\alpha_n\lambda-\beta_n\gamma)n}}\tilde{X}_k\geq 0\right)\ \geq\ \mu_{\min}^2.$$

Since the walks $X, \hat{X}$ and $\tilde{X}$ occupy disjoint parts of the environment (on the events that there are no backtrackings to the left of 0), by independence we obtain

$$\begin{aligned}\mathbb{P}\left(T_{(\alpha\lambda+\beta\gamma)n} \le n, \inf_{0\le k\le T_{(\alpha\lambda+\beta\gamma)n}} X_k \ge 0\right) \quad &\ge \quad \mathbb{P}\left(T_{\alpha_n\lambda n} \le \alpha_n n, \inf_{0\le k\le T_{\alpha_n\lambda n}} X_k \ge 0\right) \\ &\times \mathbb{P}\left(\hat{T}_{\beta_n\gamma n} \le \beta_n n, \inf_{0\le k\le \hat{T}_{\beta_n\gamma n}} \hat{X}_k \ge 0\right)\mu_{\min}^2.\end{aligned}$$

Taking logarithms of both sides of the last inequality, dividing by n, and taking the limit as $n \to \infty$ we conclude

$$\begin{aligned}\phi(\alpha\lambda+\beta\gamma) \quad \ge \quad & \lim_{n\to\infty}\frac{1}{n}\log\left(\mu_{\min}^2\right) \\ &+ \lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(T_{\alpha_n\lambda n} \le \alpha_n n, \inf_{0\le k\le T_{\alpha_n\lambda n}} X_k \ge 0\right) \\ &+ \lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\hat{T}_{\beta_n\gamma n} \le \beta_n n, \inf_{0\le k\le \hat{T}_{\beta_n\gamma n}} \hat{X}_k \ge 0\right).\end{aligned} \tag{16}$$

The first limit on the right-hand side of the last inequality is equal to 0. For the second limit we use that $\alpha_n n$ is a positive integer, hence

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(T_{\alpha_n\lambda n} \le \alpha_n n, \inf_{0\le k\le T_{\alpha_n\lambda n}} X_k \ge 0\right) \\ =\ &\lim_{n\to\infty}\frac{\alpha_n}{\alpha_n n}\log\mathbb{P}\left(T_{\alpha_n\lambda n} \le \alpha_n n, \inf_{0\le k\le T_{\alpha_n\lambda n}} X_k \ge 0\right) \\ =\ &\lim_{n\to\infty}\alpha_n \cdot \lim_{n\to\infty}\frac{1}{\alpha_n n}\log\mathbb{P}\left(T_{\alpha_n\lambda n} \le \alpha_n n, \inf_{0\le k\le T_{\alpha_n\lambda n}} X_k \ge 0\right) \\ =\ &\alpha\phi(\lambda).\end{aligned}$$

Similarly we obtain that the last term on the right-hand side of (16) is equal to $\beta\phi(\gamma)$ which completes the proof of the concavity. ☐

In a similar way we can prove that the function φ from (11) is concave in λ.

Theorem 6. *Assume that $\alpha, \beta > 0$ are real numbers such that $\alpha + \beta = 1$. Then for any λ, $\gamma > 0$ the following inequality holds*

$$\varphi(\alpha\lambda+\beta\gamma) \ge \alpha\varphi(\lambda) + \beta\varphi(\gamma).$$

Moreover, $\varphi(0) = 0$, and if the function $\xi(n)$ from (11) is the identity map $\xi(n) \equiv n$, then $\varphi(\lambda) < 0$ for $\lambda > 0$ and $\varphi(\lambda) = -\infty$ for $\lambda > L$.

Proof. The same argument used in proving Theorem 5 can be used to prove that $\varphi(0) = 0$. Also, if $\xi(n) \equiv n$ in an analogous way we can prove that $\varphi(\lambda) = -\infty$ for $\lambda > L$ and $\varphi(\lambda) \in (-\infty, 0)$ for $\lambda \in (0, L)$. The idea of the proof is very similar to the one used for Theorem 5. We start by fixing two sequences of rational numbers $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ for which $n\alpha_n$, $n\beta_n \in \mathbb{N}$, $0 \le \alpha - \alpha_n \le \frac{1}{n}$, and $0 \le \beta - \beta_n \le \frac{1}{n}$. Starting from the inclusion

$$\begin{aligned}&\left\{T_{(\alpha\lambda+\beta\gamma)\xi(n)} \le n, \inf_{0\le k\le T_{(\alpha\lambda+\beta\gamma)\xi(n)}} X_k \ge 0\right\} \\ \supseteq\ &\left\{T_{(\alpha\lambda+\beta\gamma)\xi(n)} \le n, \inf_{0\le k\le T_{(\alpha\lambda+\beta\gamma)\xi(n)}} X_k \ge 0, T_{\alpha_n\lambda\xi(n)} \le \alpha_n n\right\}\end{aligned}$$

and using the same reasoning that established the inequality (16) we obtain

$$\begin{aligned}
\varphi(\alpha\lambda+\beta\gamma) \quad \geq \quad & \lim_{n\to\infty}\frac{1}{n}\log\left(\mu_{\min}^2\right) \\
&+\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(T_{\alpha_n\lambda\xi(n)}\leq\alpha_n n, \inf_{0\leq k\leq T_{\alpha_n\lambda\xi(n)}} X_k\geq 0\right) \\
&+\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\hat{T}_{\beta_n\gamma\xi(n)}\leq\beta_n n, \inf_{0\leq k\leq \hat{T}_{\beta_n\gamma\xi(n)}} \hat{X}_k\geq 0\right).
\end{aligned} \tag{17}$$

It suffices to prove that the second limit from the right-hand side is greater than or equal to $\alpha\varphi(\lambda)$. The number $\alpha_n n$ is a positive integer and since $\alpha_n<1$ the following inequality holds $\alpha_n\xi(n)\leq\xi(\alpha_n n)$. Therefore

$$\begin{aligned}
& \lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(T_{\alpha_n\lambda\xi(n)}\leq\alpha_n n, \inf_{0\leq k\leq T_{\alpha_n\lambda\xi(n)}} X_k\geq 0\right) \\
\geq \quad & \lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(T_{\lambda\xi(\alpha_n n)}\leq\alpha_n n, \inf_{0\leq k\leq T_{\lambda\xi(\alpha_n n)}} X_k\geq 0\right) \\
= \quad & \lim_{n\to\infty}\frac{\alpha_n}{\alpha_n n}\log\mathbb{P}\left(T_{\lambda\xi(\alpha_n n)}\leq\alpha_n n, \inf_{0\leq k\leq T_{\lambda\xi(\alpha_n n)}} X_k\geq 0\right) \\
= \quad & \lim_{n\to\infty}\alpha_n\cdot\lim_{n\to\infty}\frac{1}{\alpha_n n}\log\mathbb{P}\left(T_{\lambda\xi(\alpha_n n)}\leq\alpha_n n, \inf_{0\leq k\leq T_{\lambda\xi(\alpha_n n)}} X_k\geq 0\right) \\
= \quad & \alpha\varphi(\lambda).
\end{aligned}$$

Similarly we obtain that the last term on the right-hand side of (17) is greater than or equal to $\beta\varphi(\gamma)$ which completes the proof of the concavity. □

For functions $\xi(n)$ that are not the identity map we cannot guarantee that $\varphi(\lambda)<0$. We believe this not to be true, but we do not have a proof of this.

7. Open problems

We believe that the main result, Theorem 1, holds in higher dimensions. Our proof for EDWRE differs from the proof of the analogous result for DWRE in that it requires Lemma 4, and our proof for this lemma relied heavily on the dimension being $d=1$.

Conjecture 1. *Theorem 1 holds when $d\geq 2$.*

Our proof of Lemma 4 required that $M\geq 3$ to guarantee the existence of the increasing sequence $(a_k)_{k=1}^I$ in $\mathbb{Z}$ that the walk cannot cover in time n. We believe that this technical condition can be removed, but we were only able to do so in the case $L=2$.

Conjecture 2. *Theorem 1 holds for all $M\geq 1$.*

Our next question is related to the excursions that are well understood in the case of ERW [10]. Recall that V_0^1 is the time of the first visit to 0. Since the walk starts at 0 we have that $V_0^1=0$.

Question 1. *Let $A=\{V_0^2<+\infty\}$. What is the conditional expectation $\mathbb{E}[V_0^2|A]$? How does it depend on M?*

The sequence X_n eventually gets trapped in a loop. We do not know much about the time when the walk enters the trapping loop, nor the asymptotic size of the trapping loop. It is expected that the time of entering the trapping loop depends on the number of cookies M, but it is unclear whether the size of the trapping loop does, too.

We can define the time of the entrance to the loop in the following way:

$$Z(\omega) = \inf\left\{n : \exists k \leq n, X_k = X_n, \forall j \in \{k, k+1, \dots, n\}\, L_j(\omega, X_j) \geq M - 1\right\}.$$

Question 2. *What can be said about the distribution of Z and its dependence on M? What is the length of the loop in which the walk gets stuck? Does it depend on M?*

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