34-th Austrian Mathematical Olympiad 2003
Final Round

Part 1 – May 28

1. Find all triples of prime numbers (p,q,r) such that $p^q + p^r$ is a perfct square.
2. Find the greatest and smallest value of $f(x,y) = y - 2x$, if x,y are distinct non-negative real numbers with $\dfrac{x^2+y^2}{x+y} \leq 4$.
3. Given a positive real number t, find the number of real solutions a,b,c,d of the system
$$a(1-b^2) = b(1-c^2) = c(1-d^2) = d(1-a^2) = t.$$
4. In a parallelogram $ABCD$, points E and F are the midpoints of AB and BC, respectively, and P is the intersection of EC and FD. Prove that the segments AP, BP, CP and DP divide the parallelogram into four triangles whose areas are in the ratio $1:2:3:4$.

Part 2 – June 28–29

First Day

1. Consider the polynomial $P(n) = n^3 - n^2 - 5n + 2$. Determine all integers n for which $P(n)^2$ is a square of a prime.
2. Let a,b,c be nonzero real numbers for which there exist $\alpha,\beta,\gamma \in \{-1,1\}$ with $\alpha a + \beta b + \gamma c = 0$. What is the smallest possible value of
$$\left(\frac{a^3+b^3+c^3}{abc}\right)^2?$$
3. For every lattice point (x,y) with x,y nonnegative integers, a square of side $\dfrac{0.9}{2^x 5^y}$ with center at the point (x,y) is constructed. Compute the area of the union of all these squares.

Second Day

4. Prove that, for any integer $g > 2$, there is a unique three-digit number $\overline{abc}_g$ in base g whose representation in some base $h = g \pm 1$ is $\overline{cba}_h$.
5. We are given sufficiently many stones of the forms of a rectangle 2×1 and square 1×1. Let $n > 3$ be a natural number. In how many ways can one tile a rectangle $3\times n$ using these stones, so that no two 2×1 rectangles have a common point, and each of them has the longer side parallel to the shorter side of the big rectangle?

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6. Let ABC be an acute-angled triangle. The circle k with diameter AB intersects AC and BC again at P and Q, respectively. The tangents to k at A and R meet at R, and the tangents at B and P meet at S. Show that C lies on the line RS.

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