36-th Austrian Mathematical Olympiad 2005
Final Round

Part 1 – May 30

1. Show that there exist infinitely many multiples of 2005 in which each of the decimal digits $0,1,2,\dots,9$ occurs equally many times.

2. For which integer values of a with $|a| \le 2005$ does the following system of equations have integral solutions:
$$\begin{aligned} x^2 &= y+a, \\ y^2 &= x+a? \end{aligned}$$

3. For three given real numbers a,b,c, consider $s_n = a^n + b^n + c^n$. Suppose that $s_1 = 2$, $s_2 = 6$, and $s_3 = 14$. Prove that
$$|s_n^2 - s_{n-1}s_{n+1}| = 8 \quad \text{for all integers } n > 1.$$

4. Two congruent equilateral triangles with parallel sides are given in the plane, one of them showing upwards and the other one downwards. The intersection of these triangles is a hexagon. Show that the main diagonals of this hexagon are concurrent.

Part 2 – June 8–9

First Day

1. Determine all triples (a,b,c) of positive integers such that their least common multiple equals $a+b+c$.

2. Let a,b,c,d be positive numbers. Prove the inequality
$$\frac{a+b+c+d}{abcd} \le \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}.$$

3. In an acute triangle ABC, circles k_1 and k_2 are constructed on the sides AC and BC respectively as diameters. The altitudes BE and AF intersect k_1 and k_2 respectively at points L,N and K,M, with K on segment AF and L on segment BE. Prove that $KLMN$ is a rectangle.

Second Day

4. A function f from the set $\{0,1,\dots,2005\}$ into the nonnegative integers has the property that for all (applicable) integers x,
$$f(2x+1) = f(2x), \quad f(3x+1) = f(3x), \quad f(5x+1) = f(5x).$$
At most, how many distinct values can this function take?

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5. Find all real solutions (a,b,c,d,e,f) of the system

$$\begin{aligned} 4a &= (b+c+d+e)^4, \\ 4b &= (c+d+e+f)^4, \\ &\cdots\cdots\cdots \\ 4f &= (a+b+c+d)^4. \end{aligned}$$

6. Let Q be an interior point of a cube. Show that there exist infinitely many lines g through Q such that the portion of g inside the cube is bisected by point Q.

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