14-th Bosnia and Hercegovina Mathematical Olympiad 2009

Sarajevo, May 9–10, 2009

First Day

1. Let M and N be the feet of perpendiculars from A to the external angle bisectors corresponding to the vertices B and C of $\triangle ABC$. Prove that the length of the segment MN is equal to the semi-perimeter of $\triangle ABC$.

2. Find all pairs (a,b) of natural numbers such that
$$\frac{a^2(b-a)}{b+a}$$
is a square of a prime number.

3. Let a_1, a_2, $\dots$, a_{100} be real numbers for which
$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0$$
$$a_1^2 + a_2^2 \geq 100$$
$$a_3^2 + a_4^2 + \cdots + a_{100}^2 \geq 100.$$
What is the minimal possible value for the sum $a_1 + a_2 + \cdots + a_{100}$?

Second Day

4. Given an $1 \times n$ table ($n \geq 2$), two players alternate the moves in which they write the signs $+$ and $-$ in the cells of the table. The first player always writes $+$, while the second always writes $-$. It is not allowed for two equal signs to appear in the adjacent cells. The player who can't make a move looses the game. Which of the players has a winning strategy?

5. A line intersects the sides AB and BC of $\triangle ABC$ at points M and K. If the area of the triangle MBK is equal to the area of the quadrilateral $AMKC$, prove that
$$\frac{|MB| + |BK|}{|AM| + |CA| + |KC|} \geq \frac{1}{3}.$$

6. Let n be a natural number and let $x > 0$ be a real number such that none of the numbers x, $2x$, $\dots$, nx, $\frac{1}{x}$, $\frac{2}{x}$, $\dots$, $\frac{[nx]}{x}$ is an integer. Prove that
$$[x] + [2x] + \cdots + [nx] + \left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \cdots + \left[\frac{nx}{x}\right] = n[nx].$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com