1-st Bosnia and Hercegovina Mathematical Olympiad 1996

Sarajevo, May 18–19

First Day

1. (a) If a, b, c are positive numbers, prove that for all integers $m > 0$
$$(a+b)^m + (b+c)^m + (c+a)^m \leq 2^m(a^m + b^m + c^m).$$
   (b) Does the inequality in (a) still hold if (1) a, b, c are arbitrary real numbers, or (2) m is any integer?

2. (a) Let $m \geq 2$ and n be positive integers. Prove that n divides $\phi(m^n - 1)$, where ϕ is the Euler function.
   (b) Prove that among the numbers $1, 2, \dots, n$ the number of those having the greatest common divisor with n equal to d $(d \mid n)$ is exactly $\phi(\frac{n}{d})$.

3. A point M inside a convex quadrilateral $ABCD$ is such that $ABMD$ is a parallelogram. Prove that if $\angle CBM = \angle CDM$ then $\angle ACD = \angle BCM$.

Second Day

4. Find all functions (a) $f : \mathbb{N} \to \mathbb{R}$ (b) $f : \mathbb{R} \to \mathbb{R}$ satisfying for all x, y
$$f(x+y) + f(x-y) = 2f(x)\cos y.$$

5. Ten persons were buying books. It is known that:
   (a) Every person bought four different books;
   (b) Every two persons bought at least one book in common.

   Consider the book bought by the largest number of these people. Find the smallest possible value of that number.

6. Let a and b be two fixed coprime integers and $n \leq N$ be a random positive integer (all choices are equiprobable). Find the limit (as $N \to \infty$) of the probability that the number of solutions (x, y) to the equation $ax + by = n$ in nonnegative integers equals $\left[\frac{n}{ab}\right] + 1$.

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