55-th Belarusian Mathematical Olympiad 2005

Final Round

Category C

First Day

1. Find all triples (a,b,c) of positive integers such that $abc+ab+c=a^3$.
2. Let K and M be points on the sides AB and BC respectively of a triangle ABC, and N be the intersection point of AM and CK. Assume that the quadrilaterals $AKMC$ and $KBMN$ are cyclic with the same circumradius. Find $\angle ABC$.
3. Compute $\left[\frac{2^1}{1!}+\frac{2^2}{2!}+\cdots+\frac{2^{100}}{100!}\right]$.
4. The cells of an $n\times n$ board are colored black and white, so that for any two different columns and two different rows, the four cells at their intersections are not all of the same color. Find the largest possible value of n.

Second Day

5. If m and n are positive integers, prove that $|n\sqrt{2005}-m|>\frac{1}{90n}$.
6. Suppose that there is a point K on the side CD of a trapezoid $ABCD$ with $AD \parallel BC$ such that ABK is an equilateral triangle. Show that there is a point L on the line AB such that CDL is also an equilateral triangle.
7. An infinite sequence of positive integers has the property that for any n, the product of the first n terms is divisible by their sum. Can this sequence be (a) arithmetical, (b) geometrical?
8. (a) Prove that the set $M=\{1,2,\dots,100\}$ cannot be divided into less than four classes so that whenever $a,b\in M$ and $a-b$ is a nonzero perfect square, a and b are in distinct classes.

   (b) Can M be divided into five classes with this property?

Category B

First Day

1. The altitudes BB_1 and CC_1 of an acute-angled triangle ABC intersect at H. Let l be the line through A perpendicular to AC. Prove that the lines BC, B_1C_1, and l are concurrent if and only if H is the midpoint of BB_1.

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2. A set M of nonnegative real numbers has the property that, for any two (not necessarily distinct) elements $a, b \in M$, $a+b$ is also in M. Show that if M contains a finite interval, then it also contains an infinite interval.

3. Find all positive integers n for which there exists prime numbers p, q with $q = p+2$ such that $2^n + p$ and $2^n + q$ are also prime.

4. The cells of an $(n+1) \times (n-1)$ board are painted with three colors, so that for any two different columns and two different rows, the four cells at their intersections are not all of the same color. Find the largest possible value of n.

Second Day

5. Prove that for all $n \in \mathbb{N}$, $\dfrac{1}{2n} < \{n\sqrt{7}\} < 1 - \dfrac{1}{6n}$.

6. The set $M = \{1, 2, \dots, 30\}$ is divided into k classes so that whenever a and b are distinct elements of M whose sum $a+b$ is a perfect square, a and b are in distinct classes. Find the smalest possible k.

7. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m-n+f(n)) = f(m) + f(n) \quad \text{for all } m, n \in \mathbb{N}.$$

8. Does there exist a convex heptagon such that for any of its inner angles, the angle bisector contains one of the diagonals?

Category A

First Day

1. If a and b are positive numbers, prove that

$$\left(a^2 + b + \frac{3}{4}\right)\left(b^2 + a + \frac{3}{4}\right) \geq \left(2a + \frac{1}{2}\right)\left(2b + \frac{1}{2}\right).$$

2. A line parallel to the side AC of a triangle ABC with $\angle C = 90^\circ$ intersects side AB at M and side BC at N, so that $CN : BN = AC : BC = 2 : 1$. The segments CM and AN meet at O. Let K be a point on the segment ON such that $MO + OK = KN$. The bisector of $\angle ABC$ meets the line through K perpendicular to AN at point T. Determine $\angle MTB$.

3. Find all pairs (a, b) of positive integers with $a > b$ such that $(a-b)^{ab} = a^b b^a$.

4. An $n \times n$ table is called *good* if one can paint its cells with three colors so that, for any two different rows and two different columns, the four cells at their intersections are not all of the same color.

(a) Show that there exists a good 9×9 table.

(b) Prove that if an $n \times n$ table is good, then $n < 11$.

Second Day

5. Suppose that $0 < a, b, c, d < \pi/2$ satisfy

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d =$$
$$4(\sin a \sin b \sin c \sin d - \cos a \cos b \cos c \cos d).$$

Find all possible values of $a + b + c + d$.

6. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) = f(n + f(n))$ for all $n \in \mathbb{N}$.

   (a) Prove that if the range of f is finite, then f is periodic.

   (b) Give an example of a non-periodic function f with this property.

7. The deputies in a parliament were split into 10 fractions. According to regulations, no fraction may consist of less than five people, and no two fractions may have the same number of members. After the vacation, the fractions disintegrated and several new fractions arose instead. Besides, some deputies became independent. It turned out that no two deputies that were in the same fraction before the vacation entered the same fraction after the vacation. Find the smallest possible number of independent deputies after the vacation.

8. Does there exist a convex pentagon such that for any of its inner angles, the angle bisector contains one of the diagonals?