22-nd Brazilian Mathematical Olympiad 2000

Third Round

First Day

1. A line r passes through the corner A of a sheet of paper and forms an angle α with the horizontal border. To divide the angle α into three equal parts, we perform the following construction:

   - we consider two points B, C on the vertical border such that $AB = BC$, and draw a line s through B parallel to the horizontal border;
   - we fold the sheet so as to make C and A coincide with some points C' on r and A' on s: by this, B coincides with a point B'.

   Prove that rays AA' and BB' divide the angle α into three equal parts.

2. Let $\sigma(n)$ denote the sum of all positive divisors of a positive integer n (for example, $\sigma(6) = 1+2+3+6 = 12$). We call a number n *quasi-perfect* if $\sigma(n) = 2n-1$. Let $n \bmod k$ denote the remainder of n upon division by k, and $s(n) = \sum_{k=1}^{n} (n \bmod k)$ (for example, $s(6) = 0+0+0+2+1+0 = 3$). Prove that n is quazi-perfect if and only if $s(n) = s(n-1)$.

3. Define a function f on the set of positive integers in the following way. If n is written as $2^a(2b+1)$ for integers a and b, then $f(n) = a^2+a+1$. Find the minimum positive n for which
$$f(1)+f(2)+\cdots+f(n) \geq 123456.$$

Second Day

4. The Providência avenue has an infinity of synchronized semaphores, every two consecutive ones being on the distance of 1500m. The semaphores are *on* (green light) for 1.5 minutes and *off* (red light) for 1 minute. Suppose there is a car driving through the avenue with constant speed equal to v. For what values of v can the car go arbitrarily far without stopping?

5. Let X be the set of all sequences $a = (a_1, a_2, \ldots, a_{2000})$ such that $a_i \in \{0,1,2\}$ for $1 \leq i \leq 1000$ and $a_i \in \{0,1\}$ for $1001 \leq i \leq 2000$. For $a, b \in X$, let us define the distance $d(a,b)$ as the number of indices i for which a_i and b_i are distinct. Find the number of functions $f : X \to X$ which preserve the distance, i.e. such that $d(f(a), f(b)) = d(a,b)$ for all $a, b \in X$.

6. Let C be a cube. For each of the 28 pairs of vertices of C, consider the bisector plane of the segment joining the two vertices. Determine the number of parts in which these 28 planes divide the cube.

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