23-rd Brazilian Mathematical Olympiad 2001

Third Round

First Day

1. Prove that $(a+b)(a+c) \geq 2\sqrt{abc(a+b+c)}$ for all positive real numbers a, b, c.

2. Let be given an integer $a_0 > 1$. We define a sequence $(a_n)_{n\geq 1}$ in the following way. For every $k \geq 0$, a_{k+1} is the least integer $x > a_k$ such that $(x, a_0 a_1 \cdots a_k) = 1$. Determine for which values of a_0 are all the members a_k of the sequence primes or powers of primes.

3. Let E and F be points on the side AB of a triangle ABC such that $AE = EF = FB$. Let D be the foot of perpendicular from E to line BC. Suppose that AD is perpendicular to CF and that the angles $\angle BDF$ and $\angle CFA$ are equal to x and $3x$ for some x, respectively. Calculate the ratio DB/DC.

Second Day

4. We are given a calculator with only two keys: sin and cos buttons. Initially, the display shows 1. We perform exactly 2001 operations on it, each one consisting of pressing one of the two buttons. What is the biggest result of these operations that can be obtained?

5. In a convex quadrilateral, we define an altitude as a perpendicular from the midpoint of a side to the opposite side of the quadrilateral. Prove that all the four altitudes have a common point if and only if the quadrilateral is inscribed in a circle.

6. Given a strip of cells, infinite in both directions, and n stones in the central cell (indexed by 0). The following moves are permitted.

   (A) Remove one stone from each of the cells i and $i+1$ and put one stone onto cell $i-1$;

   (B) Remove two squares from cell i and put one stone on each of the cells $i-1$, $i+2$.

   Prove that, no matter how we perform the moves, we 'll end up in a finite time with a position in which no further moves can be made. Moreover, prove that this final position does not depend on the sequence of moves.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com