24-th Brazilian Mathematical Olympiad 2002

Third Round

First Day

1. Show that there exists a set A of positive integers with the following properties:

   (a) A has 2002 elements;

   (b) The sum of any number of distinct elements of A (at least one) is never a perfect power (i.e. a number of the form a^b, where $a,b \in \mathbb{N}$ and $b \geq 2$).

2. Suppose that $ABCD$ is a convex cyclic quadrilateral and M a point on the side CD such that the triangle ADM and the quadrilateral $ABCM$ have the same area and the same perimeter. Prove that $ABCD$ has two sides of equal lengths.

3. The cells of an $m \times n$ table ($m,n \geq 2$) are numbered with numbers $1,2,\dots,mn$ in such a manner that, for each $i \leq mn-1$, the cells i and $i+1$ are adjacent (i.e. have a common side). Prove that there exists $i \leq mn-3$ such that the cells i and $i+3$ are adjacent.

Second Day

4. We define the *diameter* of a non-empty subset of $\{1,2,\dots,n\}$ as the absolute difference between its greatest element and its smallest element. Calculate the sum of the diameters of all non-empty subsets of $\{1,2,\dots,n\}$.

5. A finite number of squares with the total area 4 are given. Prove that it is possible to cover a unit square with these squares. (The squares may overlap.)

6. Arnaldo and Beatriz use smoke signals consisting of large and small clouds, to communicate during a camping. In a time before a morning coffee, Arnaldo emits a sequence of 24 clouds. Since Beatriz not always succeeds to distinguish a large cloud from a small one, Arnaldo had made a dictionary before going to the camping. The dictionary contains N sequences of length 24 (for instance, the sequence *PGPGPGPGPGPGGPGPGPGPGPGP*, where P denotes a small cloud and G denotes a large one) together with their meanings. To prevent from misinterpretations, they avoided including similar sequences in the dictionary. More precisely, any two sequences in the dictionary differ in at least 8 out of the 24 positions. Show that $N \leq 4096$.