27-th Brazilian Mathematical Olympiad 2005

Third Round

First Day – October 22

1. A natural number is *palindromic* if writing its (decimal) digits in the reverse order yields the same number. For instance, numbers 481184, 131 and 2 are palindromic. Find all pairs of positive integers (m,n) such that $\underbrace{11\dots1}_{m} \cdot \underbrace{11\dots1}_{n}$ is palindromic.

2. Determine the smallest real number C such that the inequality
$$C(x_1^{2005}+x_2^{2005}+\cdots+x_5^{2005}) \geq x_1x_2x_3x_4x_5(x_1^{125}+x_2^{125}+\cdots+x_5^{125})^{16}$$
holds for all positive real numbers x_1,x_2,x_3,x_4,x_5.

3. We say that a square is contained in a cube if all its points lie inside or on the boudary of a given cube. Find the largest $l>0$ for which there exists a square of side l that is contained in a cube of side 1.

Second Day – October 23

4. We have four charged batteries, four discharged batteries, and a radio that requires two charged batteries to work. Assume that we don't know which batteries are charged. Find the smallest number of trials that is always sufficient to make the radio work. A trial consists of picking two batteries, placing them into the radio and checking if it works.

5. Let ABC be an acute-angled triangle and F be its Fermat point (i.e. the point inside the triangle such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$). For each of the triangles ABF, BCF, CAF the Euler line (the line through the circumcenter and the centroid) is drawn. Prove that these three lines are concurrent.

6. Given positive integers a,c and an integer b, prove that there exists a positive integer x such that
$$a^x + x \equiv b \pmod{c}.$$

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