28-th Brazilian Mathematical Olympiad 2006

Third Round

First Day – October 28

1. In a triangle ABC, I is the incenter and P the intersection point of the inner bisector of $\angle B$ and AC. Prove that if $AP + AB = CB$, then triangle API is isosceles.

2. For an integer $n \geq 3$, denote by $f(n)$ the largest possible number of isosceles triangles with vertices in a set of n points in the plane with no three collinear. Show that there exist positive constants a and b such that $an^2 < f(n) < bn^2$ for all n.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(xf(y) + f(x)) = 2f(x) + xy \quad \text{for all } x, y \in \mathbb{R}.$$

Second Day – October 29

4. A positive integer is called *bold* if it has 8 positive divisors that sum up to 3240. For example, 2006 is bold because its divisors 1, 2, 17, 34, 59, 118, 1003, 2006 have the sum 3240. Find the smallest bold number.

5. Suppose that a convex 2006-gon $\mathscr{P}$ is such that its 1003 diagonals connecting the opposite vertices and the 1003 lines connecting the midpoints of opposite sides pass through a single point. Show that the opposite sides of $\mathscr{P}$ are parallel and congruent.

6. Professor Piraldo takes part in soccer matches with a lot of goals and judges a match in his own peculiar way. A match with result $m : n$ ($m \geq n$) is *tough* if $m \leq f(n)$; Here $f(n)$ is defined by $f(0) = 0$ and $f(n) = 2n - f(r) + r$ for $n \geq 1$, where r is the largest integer with $r < n$ such that $f(r) \leq n$.

   Denote $\phi = \dfrac{1+\sqrt{5}}{2}$. Prove that a match with result $m : n$ ($m \geq n$) is tough for $m \leq \phi n$ and is not tough for $m \geq \phi n + 1$.

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