29-th Brazilian Mathematical Olympiad 2007

Third Round

First Day

1. Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by $f(x)=x^2+2007x+1$. Prove that for every positive integer n the equation
$$\underbrace{f(f(\dots(f(x))\dots))}_{n\text{ times}}=0$$
has at least one solution.

2. Find the number of integers c from the set $\{-2007,-2006,\dots,2006,2008\}$ for which there exists an integer x such that
$$2^{2007}|x^2+c.$$

3. Assume n points in a plane are vertices of a convex polygon. Prove that the set of the lengths of the sides and the diagonals of the polygon has at least $[n/2]$ elements.

Second Day

4. Arnold and Bernold play the following game in 2007×2007 table: In each of his moves, Arnold paints in green four unit squares that form 2×2 square and that are not previously painted; In each of his moves, Bernold paints in red a single unit square that is not previously painted. In the beginning none of the squares is painted. Arnold starts the game and they alternate their moves. Once Arnold is not capable of making moves, Bernold continues painting all remaining squares in red. If the number of green squares is bigger than the number of red squares – the winner is Arnold. Otherwise the winner is Bernold.

   Is it possible for Bernold to win the game, no matter how Arnold plays?

5. Let $ABCD$ be a convex quadrilateral, P the intersection of the lines AB and CD, and Q the intersection of the lines AD and BC. Let O be the intersection of the diagonals AC and BD. Prove that if $\angle POQ=90^\circ$ the PO is the bisector of $\angle AOD$ and QD is the bisector of $\angle AOB$.

6. Let $x_1, x_2, \dots, x_n$ be an increasing sequence of real numbers such that every real number occurs at most twice among the differences x_j-x_i for $1\le i<j\le n$. Prove that there exists at least $[n/2]$ real numbers that occur exactly once among such differences.

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