31-st Brazilian Mathematical Olympiad 2009

Third Round

First Day

1. Emerald writes 2009^2 integers in a 2009×2009 table, one number in each cell. She sums all the numbers in each row and in each column, obtaining 4018 sums. She notices that all sums are distinct. Is it possible that all such sums are perfect squares?

2. Assume that p and q are positive prime numbers such that $q = 2p + 1$. Prove that there exists a multiple of q whose sum of digits in decimal expansion is positive and at most 3.

3. There are 2009 pebbles in some points (x, y) with both integer coordinates. An operation consists of choosing a point (a, b) with four or more pebbles, removing four pebbles from (a, b) and putting one pebble in each of the points
$$(a, b-1),\ (a, b+1),\ (a-1, b),\ (a+1, b).$$
Show that after a finite number of operations each point will have at most three pebbles. Prove that the final configuration doesn't depend on the order of the operations.

Second Day

4. Prove that there exists a positive integer n_0 with the following property: for each integer $n \geq n_0$ it is possible to partition a cube into n smaller cubes.

5. Let ABC be a triangle and) its circumcenter. The lines AB and AC meet the circumcircle of $\triangle OBC$ again at B_1 and C_1, respectively. The lines BA and BC meet the circumcircle of $\triangle OAC$ again in A_2 and C_2, respectively. The lines CA and CB meet the circumcircle of $\triangle OAB$ in A_3 and B_3, respectively. Prove that the lines A_2A_3, B_1B_3, and C_1C_2 have a common point.

6. Let $n > 3$ be a fixed integer and $x_1, x_2, \ldots, x_n$ positive real numbers. Find, in terms of n, all possible values of
$$\frac{x_1}{x_n + x_1 + x_2} + \frac{x_2}{x_1 + x_2 + x_3} + \frac{x_3}{x_2 + x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_{n-2} + x_{n-1} + x_n} + \frac{x_n}{x_{n-1} + x_n + x_1}.$$

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