Brazilian IMO & IbMO Team Selection Tests 2000

First Test – March 25, 2000

Time: 4.5 hours

1. Prove that if a, b, c are lengths of sides of a triangle and
$$2(ab^2 + bc^2 + ca^2) = a^2b + b^2c + c^2a + 3abc$$
then the triangle is equilateral.

2. For a positive integer n, let A_n be the set of all positive numbers greater than 1 and less than n which are coprime to n. Find all n such that all the elements of A_n are prime numbers.

3. Suppose that $AB \neq AC$ in a triangle ABC, and let BB', CC' be its altitudes. Let M be the midpoint of BC, H the orthocenter of ABC and D the intersection point of lines BC and $B'C'$. Prove that DH is perpendicular to AM.

4. For a positive integer n, let $V(n,b)$ be the number of decompositions of n into a product of one or more positive integers greater than b. For example, $36 = 6 \cdot 6 = 4 \cdot 9 = 3 \cdot 12 = 3 \cdot 3 \cdot 4$, so that $V(36,2) = 5$. Prove that for all positive integers n, b it holds that
$$V(n,b) < \frac{n}{b}.$$

Second Test – May 20, 2000

1. Let I be the incentre of a triangle ABC and D be the intersection point of AI and the circumcircle of ABC. Let E, F be the feet of perpendiculars from I to BD and CD, respectively. If $IE + IF = AD/2$, determine the angle $\angle BAC$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
   (i) $f(0) = 1$;
   (ii) $f(x + f(y)) = f(x + y) + 1$ for all real x, y;
   (iii) there is a rational non-integer x_0 such that $f(x_0)$ is an integer.

3. Consider an equilateral triangle with every side divided by n points into $n+1$ equal parts. We put a marker on every of the $3n$ division points. We draw lines parallel to the sides of the triangle through the division points, and this way divide the triangle into $(n+1)^2$ smaller ones.

   Consider the following game: if there is a small triangle with exactly one vertex unoccupied, we put a marker on it and simultaneously take markers from the two its occupied vertices. We repeat this operation as long as it is possible.

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(a) If $n \equiv 1 \pmod 3$, prove that we cannot manage that only one marker remains.

(b) If $n \equiv 0$ or $n \equiv 2 \pmod 3$, prove that we can finish the game leaving exactly one marker on the triangle.

4. Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there is an integer m divisible by n whose sum of digits in base 10 equals k.