Bulgarian Mathematical Olympiad 2003, III Round

First Day, 19 april 2003

1. A rectangular trapezium with area 10 and height 4 is divided with a line parallel to its bases on two trapeziums in which can be inscribed circles. Find the radiuses of these circles. (Oleg Mushkarov)

2. It is given a natural number n. Yana writes natural numbers and then Ivo deletes some of them (zero or more but not all numbers simultaneously) and then before each of the not deleted numbers inserts + or - sign, Ivo wins if the result is divisible by 2003 else Yana wins. Who of them have a winning strategy? (Ivailo Kortezov)

3. Find all real numbers a such that

$$4[an] = n + [a[an]]$$

for every natural number n. ($[x]$ is the biggest integer number not greater than x). (Nikolai Nikolov)

Second day, 20 april 2003

4. The point D from the side AC of triangle ABC is such that: $BD = CD$. Through the point E from the side BC is drawn a line parallel to BD intersecting AB at point F. If G is the intersecting point of AE and BD prove that: $\angle BCG = \angle BCF$. (Oleg Mushkarov, Nikolai Nikolov)

5. Find all real solutions of the system:

$$\begin{cases} x+y+z = 3xy \\ x^2+y^2+z^2 = 3xz \\ x^3+y^3+z^3 = 3yz \end{cases}$$

(Sava Grozdev, Svetlozar Doichev)

6. We will say that the subset C consisting of natural numbers is *good* if for each integer number k there exists $a, b \in C$, $a \neq b$ such that the numbers $a+k$ and $b+k$ aren't relative prime. Prove that if the sum of elements of C is equal to 2003 then for some $c \in C$ the set $C - \{c\}$ is also *good*.
(Aleksander Ivanov, Emil Kolev)

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