Bulgarian Mathematical Olympiad 1973, III Round

First Day

1. In a library there are 20000 books ordered on the shelves in such a way that on each of the shelves there is at least 1 and at most 199 books. Prove that there exists two shelves with same count of books of them.

   (L. Davidov)

2. Find the greatest common divisor of the numbers:

$$2^{2^2}+2^{2^1}+1, 2^{2^3}+2^{2^2}+1, \dots, 2^{2^{n+1}}+2^{2^n}+1, \dots$$

   (Hr. Lesov)

3. Find all finite sets M of whole numbers that have at least one element and have the property: for every element $x \in M$ there exists element $y \in M$ for which the following equality is satisfied: $4x^2+3 \leq 8y$. (Iv. Prodanov)

Second day

4. Prove that if n is a random natural number and α is number satisfying the condition: $0 < \alpha < \frac{\pi}{n}$, then:

$$\sin\alpha \sin 2\alpha \cdots \sin n\alpha < \frac{1}{n^n}\frac{1}{\sin^n \frac{\alpha}{2}}$$

   (L. Davidov)

5. Through the center of gravity of the triangle ABC is drawn a line intersecting the sides BC and AC in the points M and N respectively. Prove that:

$$[AMN]+[BMN] \geq \frac{4}{9}[ABC]$$

   When does equality holds? (Hr. Lesov)

6. In a sphere with radii R is inscribed a regular n-angled pyramid. The angle between two adjacent (neighboring) edges is equal to: $\frac{180^\circ}{n}$. Express the ratio between the volume and the surface of the pyramid as a function to R and n.

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