29-th Bulgarian Mathematical Olympiad 1980

Third Round

First Day

1. For a real parameter $p \neq 0$, let x_1, x_2 be the roots of the equation
$$x^2 + px - \frac{1}{2p^2} = 0.$$
Prove that $x_1^4 + x_2^4 \geq 2 + \sqrt{2}$.

2. Find all triples (a, b, c) of integers which are the sides of a triangle whose circumcircle has the diameter 6.25.

3. The sides a, b, c of an acute-angled triangle satisfy
$$c^2 b = (a+b)(a-b)^2.$$
Show that $\alpha = 3\beta$, where α and β are the angles of the triangle corresponding to sides a and b.

Second Day

4. Find all x verifying the inequality
$$\sqrt{x+1} > 1 + \sqrt{\frac{x-1}{x}}.$$

5. Points M and N are taken on the respective edges AB and CD of a tetrahedron $ABCD$ such that $AM = MB$ and $MN \perp AB$. Prove that
$$NC(BD^2 - AD^2) = ND(AC^2 - BC^2).$$

6. Prove that if the squares of edge lengths of a rectangular parallelepiped are integers and its main diagonal is $\sqrt{73}$, then its volume does not exceed 120.

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