Bulgarian Mathematical Olympiad 1969, IV Round

First Day

1. If the sum of x^5, y^5 and z^5, where x, y and z are integer numbers, is divisible by 25 then the sum of some two of them is divisible by 25.

2. Prove that
$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2$$
for every $n \in \mathbb{N}$.

3. Some of the points in the plane are *white* and other are *blue* (every point from the plane is white or blue). Prove that for every positive number r:
   (a) there are at least two points with different color and the distance between them is equal to r;
   (b) there are at least two points with the same color and the distance between them is equal to r;
   (c) will the statements above be true if the *plane* is replaced with the word *line*?

Second day

4. Find the sides of the triangle if it is known that the inscribed circle meets one of its medians in two points and these points divide the median to three equal segments and the area of the triangle is equal to $6\sqrt{14}$ cm^2.

5. Prove the equality:
$$\prod_{k=1}^{2m} \cos \frac{k\pi}{2m+1} = \frac{(-1)^m}{4m}$$

6. It is given that $r = \left[3\left(\sqrt{6}-1\right) - 4\left(\sqrt{3}+1\right) + 5\sqrt{2}\right] R$ where r and R are radii of the inscribed and circumscribed spheres in the regular n-angled pyramid. If it is known that the centers of the spheres given coincides:
   (a) find n;
   (b) if $n = 3$ and the lengths of all edges are equal to a find the volumes of the parts from the pyramid after drawing a plane μ, which intersects two of the edges passing through point A respectively in the points E and F in such a way that $|AE| = p$ and $|AF| = q$ ($p < a$, $q < a$), intersects the extension of the third edge behind opposite of the vertex A wall in the point G in such a way that $|AG| = t$ ($t > a$).

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