Bulgarian Mathematical Olympiad 1970, IV Round

First Day

1. Find all natural numbers $a > 1$, with the property: every prime divisor of $a^6 - 1$ divides also at least one of the numbers $a^3 - 1$, $a^2 - 1$.

   (7 Points, K. Dochev)

2. Two bicyclists traveled the distance from A to B, which is 100 Km with speed 30 Km/h and it is known that the first is started 30 minutes before the second. 20 minutes after the start of the first bicyclistfrom A is started a control car which speed is 90 Km/h and it is known that the car is reached the first bicyclist and is driwing together with him 10 minutes, went back to the second and was driving 10 minutes with him and after that the car is started again to the first bicyclist with speed 90 Km/h and etc. to the end of the distance. How many times the car were drive together with the first bicyclist? (5 Points, K. Dochev)

3. Over a chessboard (with 64 squares) are situated 32 white and 32 black pools. We say that two pools form a mixed pair when they are with different colors and lies on one and the same row or column. Find the maximum and the minimum of the mixed pairs for all possible situations of the pools.

   (8 Points, K. Dochev)

Second day

4. Let $\delta_0 = \triangle A_0B_0C_0$ is a triangle with vertices A_0, $B_=0$, C_0. Over each of the side B_0C_0, C_0A_0, A_0B_0 are constructed squares in the halfplane, not containing the respective vertex A_0, B_0, C_0 and A_1, B_1, C_1 are the centers of the constructed squares. If we use the triangle $\delta_1 = \triangle A_1B_1C_1$ in the same way we may construct the triangle $\delta_2 = \triangle A_2B_2C_2$; from $\delta_2 = \triangle A_2B_2C_2$ we may construct $\delta_3 = \triangle A_3B_3C_3$ and etc. Prove that:

   (a) segments A_0A_1, B_0B_1, C_0C_1 are respectively equal and perpendicular to B_1C_1, C_1A_1, A_1B_1;

   (b) vertices A_1, B_1, C_1 of the triangle δ_1 lies respectively over the segments A_0A_3, B_0B_3, C_0C_3 (defined by the vertices of δ_0 and δ_1) and divide them in ratio 2:1. (7 Points, K. Dochev)

5. Prove that for $n \geq 5$ the side of regular inscribed in a circle n-gon is bigger than the side of regular circumscribed around the same circle $n+1$-gon and if $n \leq 4$ is true the opposite statement. (6 Points)

6. In the space are given the points A, B, C and a sphere with center O and radii 1. Find the point X from the sphere for which the sum $f(X) = |XA|^2 + |XB|^2 + |XC|^2$ attains its maximal and minimal value. ($|XA|$ is the distance from X to A, $|XB|$ and $|XC|$ are defined by analogy). Prove that if the segments OA, OB, OC are mutually perpendicular and d is the distance from the center O to the center of gravity of the triangle ABC then:

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(a) the maximum of $f(X)$ is equal to $9d^2+3+6d$;

(b) the minimum of $f(X)$ is equal to $9d^2+3-6d$.

(7 Points, K. Dochev and I. Dimovski)

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