32-nd Bulgarian Mathematical Olympiad 1983

Fourth Round

First Day

1. Determine all natural numbers n for which there exists a permutation $(a_1, a_2, \dots, a_n)$ of the numbers $0, 1, \dots, n-1$ such that, if b_i is the remainder of $a_1 a_2 \cdots a_i$ upon division by n for $i = 1, \dots, n$, then $(b_1, b_2, \dots, b_n)$ is also a permutation of $0, 1, \dots, n-1$.

2. Let $b_1 \geq b_2 \geq \cdots b_n$ be nonnegative numbers, and $(a_1, a_2, \dots, a_n)$ be an arbitrary permutation of these numbers. Prove that for every $t \geq 0$,
$$(a_1a_2+t)(a_3a_4+t)\cdots(a_{2n-1}a_{2n}+t) \leq (b_1b_2+t)(b_3b_4+t)\cdots(b_{2n-1}b_{2n}+t).$$

3. A regular triangular pyramid $ABCD$ with the base side $AB = a$ and the lateral edge $AD = b$ is given. Let M and N be the midpoints of AB and CD respectively. A line α through MN intersects the edges AD and BC at P and Q, respectively.
   (a) Prove that $AP/AD = BQ/BC$.
   (b) Find the ratio AP/AD which minimizes the area of $MQNP$.

Second Day

4. Find the smallest possible side of a square in which five circles of radius 1 can be placed, so that no two of them have a common interior point.

5. Can the polynomials $x^5 - x - 1$ and $x^2 + ax + b$, where a and b are rational numbers, have common complex roots?

6. Suppose that a, b, c are real numbers such that $[an] + [bn] = [cn]$ for every natural number n. Prove that at least one of the numbers a, b is an integer.

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