34-th Bulgarian Mathematical Olympiad 1985

Fourth Round – May

First Day

1. Let $f(x)$ be a non-constant polynomial with integer coefficients and n,k be natural numbers. Show that there exist n consecutive natural numbers $a,a+1,\dots,a+n-1$ such that the numbers $f(a),f(a+1),\dots,f(a+n-1)$ all have at least k prime factors. (We say that the number $p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ has $\alpha_1+\cdots+\alpha_s$ prime factors.)

2. Find all real parameters a for which all the roots of the equation

$$x^6+3x^5+(6-a)x^4+(7-2a)x^3+(6-a)x^2+3x+1$$

are real.

3. A pyramid $MABCD$ with the top-vertex M is circumscribed about a sphere with center O so that O lies on the altitude of the pyramid. Each of the planes ACM,BDM,ABO divides the lateral surface of the pyramid into two parts of equal areas. The areas of the sections of the planes ACM and ABO inside the pyramid are in ratio $(\sqrt{2}+2):4$. Determine the angle δ between the planes ACM and ABO, and the dihedral angle of the pyramid at the edge AB.

Second Day

4. Seven points are given in space, no four of which are on a plane. Each of segments with the endpoints in these points is painted black or red. Prove that there are two monochromatic triangles (not necessarily both of the same color) with no common edge. Does the statement hold for six points?

5. Let P be a point on the median CM of a triangle ABC with $AC\neq BC$ and the acute angle γ at C, such that the bisectors of $\angle PAC$ and $\angle PBC$ intersect at a point Q on the median CM. Determine $\angle APB$ and $\angle AQB$.

6. Let α_a denote the greatest odd divisor of a natural number a, and let $S_b=\sum_{a=1}^{b}\frac{\alpha_a}{a}$. Prove that the sequence S_b/b has a finite limit when $b\to\infty$, and find this limit.

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