36-th Bulgarian Mathematical Olympiad 1987

Fourth Round – Sofia, May 16-17

First Day

1. Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ $(n \geq 3)$ be a polynomial with real coefficients and n real roots, such that $a_{n-1}/a_n > n+1$. Prove that if $a_{n-2} = 0$, then at least one root of $f(x)$ lies in the open interval $(-\frac{1}{2}, \frac{1}{n+1})$.

2. Let be given a polygon P which is mapped onto itself by two rotations: ρ_1 with center O_1 and angle ω_1, and ρ_2 with center O_2 and angle ω_2 $(0 < \omega_i < 2\pi)$. Show that the ratio ω_1/ω_2 is rational.

3. Let $MABCD$ be a pyramid with the square $ABCD$ as the base, in which $MA = MD$, $MA^2 + AB^2 = MB^2$ and the area of $\triangle ADM$ is equal to 1. Determine the radius of the largest ball that is contained in the given pyramid.

Second Day

4. The sequence $(x_n)_{n \in \mathbb{N}}$ is defined by $x_1 = x_2 = 1$, $x_{n+2} = 14x_{n+1} - x_n - 4$ for each $n \in \mathbb{N}$. Prove that all terms of this sequence are perfect squares.

5. Let E be a point on the median AD of a triangle ABC, and F be the projection of E onto BC. From a point M on EF the perpendiculars MN to AC and MP to AB are drawn. Prove that if the points N, E, P lie on a line, then M lies on the bisector of $\angle BAC$.

6. Let Δ be the set of all triangles inscribed in a given circle, with angles whose measures are integer numbers of degrees different than $45^\circ, 90^\circ$ and 135°. For each triangle $T \in \Delta$, $f(T)$ denotes the triangle with vertices at the second intersection points of the altitudes of T with the circle.

   (a) Prove that there exists a natural number n such that for every triangle $T \in \Delta$, among the triangles $T, f(T), \dots, f^n(T)$ (where $f^0(T) = T$ and $f^k(T) = f(f^{k-1}(T))$) at least two are equal.

   (b) Find the smallest n with the property from (a).

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com