44-th Bulgarian Mathematical Olympiad 1995

Fourth Round

First Day

1. Find the number of integers $n > 1$ which divide $a^{25} - a$ for every integer a.

2. Let p be the semiperimeter of a triangle ABC. Points E and F are taken on line AB such that $CE = CF = p$. Prove that the circumcircle k of $\triangle EFC$ is tangent to the excircle k_1 of $\triangle ABC$ corresponding to AB.

3. Two players A and B take stones one after the other from a heap with $n \geq 2$ stones. A begins the game and takes at least one stone, but no more than $n-1$ stones. Thereafter, a player on turn takes at least one, but no more than the other player has taken before him. The player who takes the last stone wins. Who of the players has a winning strategy?

Second Day

4. Points A_1, B_1, C_1 are selected on the sides BC, CA, AB respectively of an equilateral triangle ABC in such a way that the inradii of the triangles C_1AB_1, A_1BC_1, B_1CA_1 and $A_1B_1C_1$ are equal. Prove that A_1, B_1, C_1 are the midpoints of the corresponding sides.

5. Let $A = \{1, 2, \dots, m+n\}$, where m, n are positive integers, and let the function $f : A \to A$ be defined by:
$$f(m) = 1, \;\; f(m+n) = m+1 \;\text{ and }\; f(i) = i+1 \;\text{ for all the other } i.$$
   (a) Prove that if m and n are odd, then there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.
   (b) Prove that if m is even, then there is a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$ is and only if $n = m$.

6. Suppose that x and y are different real numbers such that $\dfrac{x^n - y^n}{x - y}$ is an integer for some four consecutive positive integers n. Prove that $\dfrac{x^n - y^n}{x - y}$ is an integer for all positive integers n.

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