47-th Bulgarian Mathematical Olympiad 1998

Fourth Round – May 16–17, 1998

First Day

1. Let n be a natural number. Find the least natural number k for which there exist k sequences of 0 and 1 of length $2n+2$ with the following property: any sequence of 0 and 1 of length $2n+2$ coincides with some of these k sequences in at least $n+2$ positions.

2. The polynomials $P_n(x,y)$ ($n \in \mathbb{N}$) are defined by $P_1(x,y) = 1$ and
$$P_{n+1}(x,y) = (x+y-1)(y+1)P_n(x,y+2) + (y-y^2)P_n(x,y), \quad n \geq 1.$$
Prove that $P_n(x,y) = P_n(y,x)$ for all x,y and n.

3. On the sides of a non-obtuse triangle ABC a square, a regular n gon and a regular m-gon ($m,n > 5$) are constructed externally, so that their centers are vertices of a regular triangle. Prove that $m = n = 6$ and find the angles of $\triangle ABC$.

Second Day

4. Let $a_1, a_2, \dots, a_n$ be real numbers, not all zero. Prove that the equation
$$\sqrt{1+a_1x} + \sqrt{1+a_2x} + \cdots + \sqrt{1+a_nx} = n$$
has at most one nonzero real root.

5. Suppose that m and n are natural numbers such that
$$A = \frac{(m+3)^n + 1}{3m}$$
is an integer. Prove that A must be odd.

6. The sides and diagonals of a regular n-gon $\mathscr{X}$ are colored in k colors so that:

   (i) For each color a and ant two vertices A,B of $\mathscr{X}$, the segment AB is of color a or there is a vertex C such that AC and BC are of color a;

   (ii) The sides of any triangle with vertices at vertices of $\mathscr{X}$ are colored in at most two colors.

   Prove that $k \leq 2$.

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