Bulgarian Team Selection Tests 2007

Selection Tests for Balkan MO

First Test - Sofia, April 7

1. Consider a triangle ABC with $\angle A = 30^\circ$ and the circumradius 1. For any point X inside the triangle or on its boundary denote $m(X) = \min\{AX, BX, CX\}$. Find the angles of the triangle if the maximum value of $m(X)$ equals $\sqrt{3}/3$.

2. Find all $a \in \mathbb{R}$ for which there exists a nonconstant function $f : (0,1] \to \mathbb{R}$ satisfying
$$a + f(x+y-xy) + f(x)f(y) \le f(x) + f(y) \quad \text{for any } x, y \in (0,1].$$

3. In a scalene triangle ABC, I is the incenter and AI and BI meet the opposite sides at A_1 and B_1 respectively. The lines through A_1 and B_1 parallel to AC and BC respectively meet the line CI at A_2 and B_2. Lines AA_2 and BB_2 meet at N, and M is the midpoint of AB. If $CN \parallel IM$, find the ratio $CN : IM$.

4. Let x be a vertex of a non-oriented graph G. The transformation φ_x of G consists of deleting all edges incident with x and drawing edges xy for all vertices y that were not joined to x by an edge. A graph H is said to be G-*obtainable* if it can be obtained from G by a sequence of transformations of the above form.
Let n be a positive integer divisible by 4. Prove that for every graph G with $4n$ vertices and n edges there exists a G-obtainable graph containing at least $9n^2/4$ triangles.

Second Test – Sofia, April 9

1. In a triangle ABC with $AC = BC$, point M on side AB is such that $AM = 2MB$, F is the midpoint of BC and H the foot of the perpendicular from M to AF. Prove that $\angle BHF = \angle ABC$.

2. Let n, k be integers with $n \ge 2k > 3$ and $A = \{1, 2, \dots, n\}$. Find all values of n and k for which the number of k-element subsets of A is $2n-k$ times that of two-element subsets of A.

3. Given an integer $n \ge 2$, find the largest constant $C(n)$ for which the inequality
$$\sum_{i=1}^{n} x_i \ge C(n) \sum_{1 \le j < i \le n} (2x_i x_j + \sqrt{x_i x_j})$$
holds for all real numbers $x_i \in (0,1)$ satisfying $(1-x_i)(1-x_j) \ge \frac{1}{4}$ for $1 \le j < i \le n$.

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4. Let p be a prime number of the form $4k+3$ ($k \in \mathbb{N}_0$). For any two integers x, y not divisible by p, denote by $f(x,y)$ the remainder of $(x^2+y^2)^2$ in division by p. How many different values can f take?

Selection Tests for IMO

First Test – May 16-17

1. The sequence $(a_n)_{n=1}^{\infty}$ is such that $a_1 > 0$ and $a_{n+1} = \dfrac{a_n}{1+a_n^2}$ for $n \geq 1$.

   (a) Prove that $a_n \leq \frac{1}{\sqrt{2n}}$.

   (b) Show that there exists n for which $a_n > \frac{7}{10\sqrt{n}}$.

2. In a convex quadrilateral $A_1A_2A_3A_4A_5$ the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5$, $A_4A_5A_1$, $A_5A_1A_2$ have the same area. Prove that there exists a point M in the plane such that the triangles A_1MA_2, A_2MA_3, A_3MA_4, A_4MA_5, A_5MA_1 also have the same area.

3. Prove that there are no distinct positive integers x and y such that
$$x^{2007} + y! = y^{2007} + x!.$$

4. Let P be a point on side AB of a triangle ABC. Consider all pairs of points $X \in BC$, $Y \in AC$ such that $\angle PXB = \angle PYA$. Prove that the midpoints of all such segments XY lie on a single line.

5. Real numbers a_i, b_i ($1 \leq i \leq n$) satisfy $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$ and $\sum_{i=1}^{n} a_i b_i = 0$. Prove that
$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq n.$$

6. Denote by $\mathscr{P}(S)$ the family of all subsets of a finite set S (including the empty set and S itself). The function $f : \mathscr{P}(S) \to \mathbb{R}$ satisfies
$$f(X \cap Y) = \min\{f(X), f(Y)\} \quad \text{for any } X, Y \in \mathscr{P}(S).$$
Find the largest number of distinct values which f can take.

Second Test – May 26-27

1. Two circles k_1 and k_2 with centers O_1 and O_2 respectively are externally tangent at P. A circle k_3 is tangent to k_1 at Q and to k_2 at R. The lines PQ and PR meet k_3 again at A and B, respectively. If AO_2 and BO_1 intersect at point S, prove that $SP \perp O_1O_2$.

2. Find all positive integers m for which

$$\frac{2^m\alpha^m-(\alpha+\beta)^m-(\alpha-\beta)^m}{3\alpha^2+\beta^2}$$

is an integer for all nonzero integers α and β.

3. Find all integers $n\geq 3$ such that for any two positive integers $m,r<n-1$ there exist m distinct elements of the set $\{1,2,\dots n-1\}$ whose sum is congruent to r modulo n.

4. Solve the system

$$x^2+yu=(x+u)^n,\quad x^2+yz=u^4,$$

where x,y,z are prime numbers and u a positive integer.

5. Find all pairs of functions $f,g:\mathbb{R}\to\mathbb{R}$ such that

   (a) $f(xg(y+1))+y=xf(y)+f(x+g(y))$ for any $x,y\in\mathbb{R}$ and

   (b) $f(0)+g(0)=0$.

6. Show that $n=11$ is the smallest positive integer such that for any coloring of the edges of a complete graph of n vertices with three colors there exists a monochromatic cycle of length 4.