3rd Canadian Mathematical Olympiad 1971

1. Let DB be a chord of a circle with center O and E a point on DB such that $DE = 3$ and $EB = 5$. The ray OE cuts the circle at C. Given $EC = 1$, find the radius of the circle.

2. Let x, y be positive real numbers such that $x + y = 1$. Prove that
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) \geq 9$$

3. Let $ABCD$ be a quadrilateral such that $AD = BC$. Show that if $\angle ADC > \angle BCD$ then $AC > BD$.

4. Find all real values of a for which the polynomials $x^2 + ax + 1$ and $x^2 + x + a$ have at least one common root.

5. Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a polynomial with integer coefficients. If $p(0)$ and $p(1)$ are both odd, show that $p(x)$ has no integral roots.

6. Prove that $n^2 + 2n + 12$ is not divisible by 121 for any $n \in \mathbb{N}$.

7. Let n be a five-digit integer (whose first digit is non-zero) and m be the four-digit number formed from n by deleting its middle digit. Find all n such that n/m is an integer.

8. A regular pentagon is inscribed in a circle of radius r. From any point P inside the pentagon, perpendiculars are dropped to the sides of the pentagon (or extensions thereof).
   (a) Prove that the sum of the lengths of these perpendiculars is constant.
   (b) Express this constant in terms of r.

9. Two flag poles of heights h and k are situated $2a$ units apart on a level surface. Describe the set of points of the surface at which the angles of elevation of the tops of the poles are equal.

10. Suppose that n people each know one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything? Justify your answer.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com