16-th Canadian Mathematical Olympiad 1984

May 2, 1984

1. Prove that the sum of the squares of 1984 consecutive positive integers cannot be a perfect square.

2. Alice and Bob are in a hardware store. The store sells colored sleeves that fir over keys to distinguish them. The following conversation takes place:

   Alice: Are you going to cover your keys?

   Bob: I would like to, but there are only 7 colors and I have 8 keys.

   Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

   Bob: You must be careful what you mean by ”next to” or ”three keys over from” since you can turn the key ring over and the keys are arranged in a circle.

   Alice: Even so, you don’t need 8 colors.

   What is the smallest number of colors needed to distinguish n keys if all the keys are to be covered?

3. An integer is said to be *digitally divisible* if

   (i) none of its digits is zero, and

   (ii) it is divisible by the sum of its digits (e.g. 322 is digitally divisible).

   Prove that there are infinitely many digitally divisible integers.

4. An acute-angled triangle has area 1. Show that there is a point inside the triangle whose distance from each of the vertices is at least $2/\sqrt[4]{27}$.

5. Prove that among any 7 real numbers there exist two, say x and y, such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$