Chinese IMO Team Selection Test 1997

Time: 4.5 hours each day.

First Day – April 1

1. Let be given a triangle ABC and a real number $t > 1$. For a point P on the arc BAC of the circumcircle of the triangle, let U and V be the points on the rays BP and CP respectively such that $BU = t \cdot BA$ and $CV = t \cdot CA$, and let Q be the point on the ray UV such that $UQ = t \cdot UV$. Find the locus of Q as P describes the arc BAC.

2. In a football championship n teams take part and every two play one match. The winner of a match gets 3 points, the loser gets no points; in the case of a draw, both teams get 1 point. How many points at least should a team get to be sure that at most $k-1$ other teams have at least as good score?

3. Determine $m \in \mathbb{N}$ for which there exists a sequence of integers x_n with the following properties:

   (i) $x_0 = 1$ and $x_1 = 337$;

   (ii) For each $n \in \mathbb{N}$, $x_{n+1}x_{n-1} - x_n^2 + \frac{3}{4}(x_{n+1} + x_{n-1} - 2x_n) = m$;

   (iii) $\dfrac{(x_n+1)(2x_n+1)}{6}$ is a perfect square for each n.

Second Day – April 2

4. Find all polynomials $F(x) = \sum_{i=0}^{n} a_{2i}x^{2n-2i}$ with the following properties:

   (i) All roots of F are purely imaginary;

   (ii) It holds that $\sum_{i=0}^{n} a_{2i}a_{2n-2i} \leq \binom{2n}{n} a_0 a_{2n}$.

5. Let $A_1, A_2, \dots, A_m$ be 5-element subsets of the set $\{1, 2, \dots, n\}$, where $n \geq 6$. If

$$m > \frac{n(n-1)(n-2)(n-3)(4n-15)}{600},$$

   prove that there exist indices $1 \leq i_1 < i_2 < \cdots < i_6 \leq m$ such that

$$\left| \bigcup_{j=1}^{6} A_{i_j} \right| = 6.$$

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6. There are 1997 pieces of medicine and three bottles A, B, C which can contain at most 1997, 97, 17 pieces of medicine respectively. Initially, all the 1997 pieces are placed in bottle A, and the three bottles are closed. Every piece contains 100 parts. When a bottle is opened, all the pieces in the bottle lose one part. A man wants to take all the medicines, but every day he can only open some bottles, take one piece, move some pieces between the bottles, and close them. How many parts of the pieces will he lose at least?

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