Chinese IMO Team Selection Test 1999

Time: 4.5 hours each day.

First Day

1. Let $x_1, x_2, \dots, x_n$ be positive reals whose sum equals 1. Find the maximum possible value of $\sum_{i=1}^{n}(x_i^4 - x_i^5)$.

2. Find all prime numbers p with the property that, for all primes q, the remainer of p upon division by q is squarefree (i.e. not divisible by any square greater than 1).

3. Find the least n for which there exist n subsets $A_1, A_2, \dots, A_n$ of set $S = \{1, 2, \dots, 15\}$ satisfying:

   (i) $|A_i| = 7$ for all i;

   (ii) $|A_i \cap A_j| \leq 3$ for any two distinct i, j;

   (iii) for any 3-element subset $M \subset S$ there is an A_k containing M.

Second Day

4. Let a circle touch the sides AB, BC of a convex quadrilateral $ABCD$ at G and H and intersect AC at E and F. Find the condition $ABCD$ must satisfy in order to exist a circle passing through E, F and touching DA, DC.

5. Let m be an even positive integer.

   (a) Show that there exist integers $x_1, x_2, \dots, x_{2m}$ such that $x_i x_{i+m} = x_{i+1} x_{i+m-1} + 1$ for $i = 1, 2, \dots, m-1$.

   (b) Prove that, if $x_1, x_2, \dots, x_{2m}$ satisfy (a), one can construct an infinite sequence $(y_k)_{k\in\mathbb{Z}}$ of integers such that $y_i = x_i$ for $i = 1, \dots, 2m$ and $y_k y_{k+m} = y_{k+1} y_{k+m-1} + 1$ for all integers k.

6. For all permutations $\tau = (x_1, \dots, x_{10})$ of numbers $1, 2, \dots, 10$, define

$$S(\tau) = \sum_{i=1}^{10} |2x_i - 3x_{i+1}|$$

(where $x_{11} = x_1$). Find the maximum and minimum values of $S(\tau)$ and all the permutations τ for which those are attained.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com