49-th Czech and Slovak Mathematical Olympiad 2000

Third Round – Bilovec, April 9-12, 2000

Category A

1. Let n be a natural number. Prove that the number $4 \cdot 3^{2^n} + 3 \cdot 4^{2^n}$ is divisible by 13 if and only if n is even. *(J. Šimša)*

2. Let be given an isosceles triangle ABC with the base AB. A point P is chosen on the altitude CD so that the incircles of ABP and $PECF$ are congruent, where E and F are the intersections of AP and BP with the opposite sides of the triangle, respectively. Prove that the incircles of triangles ADP and BCP are also congruent. *(J. Šimša, K. Horák)*

3. In the plane are given 2000 congruent triangles of area 1, which are all images of one triangle under translations. Each of these triangles contains the centroid of every other triangle. Prove that the union of these triangles has area less than $\frac{22}{9}$. *(P. Calábek)*

4. For which quadratic polynomials $f(x)$ does there exist a quadratic polynomial $g(x)$ such that the equations $g(f(x)) = 0$ and $f(x)g(x) = 0$ have the same roots, which are mutually distinct and form an arithmetic progression? *(P. Černek)*

5. Monika made a paper model of a tetrahedron whose base is a right-angled triangle. When she cut the model along the legs of the base and the median of a lateral face corresponding to one of the legs, she obtained a square of side a. Compute the volume of the tetrahedron. *(P. Leischner)*

6. Find all four-digit numbers $\overline{abcd}$ (in decimal system) such that
$$\overline{abcd} = (\overline{ac}+1)\left(\overline{bd}+1\right).$$ *(J. Zhouf)*