53-rd Czech and Slovak Mathematical Olympiad 2004

Third Round – Přerov, March 28-31, 2004

Category A

1. Find all triples (x,y,z) of real numbers satisfying:
$$x^2+y^2+z^2 \leq 6+\min\left\{x^2-\frac{8}{x^4},\, y^2-\frac{8}{y^4},\, z^2-\frac{8}{z^4}\right\}$$
(J. Švrček)

2. For an arbitrary positive integer n consider all possible words of n letters A and B and denote by p_n the number of those words containing neither $AAAA$ nor BBB. Calculate the value of
$$\frac{p_{2004}-p_{2002}-p_{1999}}{p_{2001}+p_{2000}}.$$
(R. Kučera)

3. In the plane are given a circle k and 121 lines $p_1, p_2, \dots, p_{121}$ intersecting k. On each p_i a point A_i interior to k is selected. Prove that there exists a point X on k such that lines A_iX and p_i form an angle smaller than 21° for at least 29 different indices i. *(J. Šimša)*

4. Find all natural numbers n for which $\frac{n}{1!}+\frac{n}{2!}+\cdots+\frac{n}{n!}$ is an integer. *(E. Kováč)*

5. Let L be an arbitrary point on the shorter arc CD of the circumcircle of a square $ABCD$. Let K be the intersection of AL and CD, M be the intersection of AD and CL, and N be the intersection of MK and BC. Prove that points B,L,M,N lie on a circle. *(J. Švrček)*

6. Find all functions $f:\mathbb{R}^+\to\mathbb{R}+$ such that for any $x,y>0$,
$$x^2\left(f(x)+f(y)\right)=(x+y)f\left(f(x)y\right).$$
(P. Kaňovský)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com