43-rd Czech and Slovak Mathematical Olympiad 1994

1. Let $f : \mathbb{N} \to \mathbb{N}$ be a function which satisfies
$$f(x) + f(x+2) \le 2f(x+1) \quad \text{for any } x \in \mathbb{N}.$$
Prove that there exists a line in the coordinate plane containing infinitely many points of the form $(n, f(n))$, $n \in \mathbb{N}$.

2. A cube of volume V contains a convex polyhedron M. The orthogonal projection of M onto each face of the cube covers the entire face. What is the smallest possible volume of polyhedron M?

3. A convex 1994-gon M is given in the plane. A closed polygonal line consists of 997 of its diagonals. Each diagonal divides M into two sides, and the smaller of the numbers of edges on the two sides of M is defined to be the *length* of the diagonal. Is it posible to have
   (a) 991 diagonals of length 3 and 6 of length 2?
   (b) 985 diagonals of length 6, 4 of length 8, and 8 of length 3?

4. Let $a_1, a_2, \dots$ be a sequence of natural numbers such that for each n, the product $(a_n - 1)(a_n - 2)\cdots(a_n - n^2)$ is a positive integral multiple of n^{n^2-1}. Prove that for any finite set P of prime numbers the following inequality holds:
$$\sum_{p\in P} \frac{1}{\log_p a_p} < 1.$$

5. In an acute-angled triangle ABC, the altitudes AA_1, BB_1, CC_1 intersect at point V. If the triangles AC_1V, BA_1V, CB_1V have the same area, does it follow that the triangle ABC is equilateral?

6. Show that from any four distinct numbers lying in the interval $(0,1)$ one can choose two distinct numbers a and b such that
$$\sqrt{(1-a^2)(1-b^2)} > \frac{a}{2b} + \frac{b}{2a} - ab - \frac{1}{8ab}.$$

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