34-th German Federal Mathematical Competition 2003/04

Second Round

1. Let k be a positive integer. We call an integer *k-typical* if each of its divisors leaves a remainder 1 upon division by k. Prove that:

   (a) If the number of divisors of a positive integer n (including 1 and n) is k-typical then n is a k-th power of an integer.

   (b) The converse of (a) is false if k is greater than 2.

2. Finitely many chords are drawn in a circle of radius 1. Every diameter has common points with at most k of these chords, where k is a given positive integer. Prove that the sum of the lengths of all the chords is less than $k\pi$.

3. Two circles k_1 and k_2 intersect at distinct points A and B. The tangents to k_2 and k_1 at A meet k_1 and k_2 respectively again in C_1 and C_2. The line C_1C_2 intersects k_1 again in D. Prove that BD bisects the chord AC_2.

4. Prove that there exist infinitely many pairs of distinct positive rational numbers (x, y) such that both $\sqrt{x^2+y^3}$ and $\sqrt{x^3+y^2}$ are rational numbers.

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