35-th German Mathematical Olympiad 1996

4-th Round – Hamburg

Grade 10

First Day

1. The cells of a chessboard 4×4 are denoted as $a1, \dots, d4$ in the usual manner. A knight standing at $a1$ has to reach $d4$ in exactly n moves, not visiting any cell more than once. Find all n for which this is possible.

2. Alex and Beate are trying to solve the following problem. For given $n \in \mathbb{N}$, their task is to place n congruent circles, as large as possible, inside a given square of side a so that no two of them overlap.

   For $n = 6$, Alex made the placement as shown on fig.1, while Beate made the placement on fig.2. Who of them used larger circles?

3. A pupil wants to construct a triangle ABC, given the length $c = AB$, the altitude h_c from C and the angle $\varepsilon = \alpha - \beta$. Here c and h_c are arbitrary and ε satisfies $0 < \varepsilon < 90^\circ$.

   (a) Is there such a triangle for any c, h_c and ε?

   (b) Is this triangle unique up to the congruence?

   (c) Show how to construct one such triangle, if it exists.

Second Day

4. Consider all 10-digit integers in which each of the digits $0, 1, \dots, 9$ appears exactly once. Show that at least 50000 among these numbers are divisible by 11.

5. We look for a series $(a_1, a_2, \dots, a_n)$ of consecutive natural numbers with the property that none of the a_i has the sum of digit divisible by 5.

   (a) What is the largest n for which such a sequence exists?

   (b) How many such series with the maximal n are there within the range $1, \dots, 1000$?

6. A student selected two points X, Y on a segment AB and constructed the squares $AXPQ$ and $XBRS$ on one side of AB, and $AYTU$ and $YBVW$ on the other side. Then he denoted the centers of these squares by K, L, M, N respectively. He assumed that the segments KM and LN are mutually perpendicular and of equal lengths. Does this statement always hold?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

Grades 11-13

First Day

1. Find all natural numbers n with the following property: Given the decimal writing of n, adding a few digits one can obtain the decimal writing of $1996n$.

2. Let a and b be positive real numbers smaller than 1. Prove that the following two statements are equivalent:

   (i) $a+b=1$;

   (ii) Whenever x, y are positive real numbers such that $x<1$, $y<1$, $ax+by<1$, the following inequlity holds:

$$\frac{1}{1-ax-by} \leq \frac{a}{1-x}+\frac{b}{1-y}.$$

3. Let be given an arbitrary tetrahedron $ABCD$ with volume V. Consider all lines which pass through the barycenter S of the tetrahedron and intersect the edges AD, BD, CD at points A', B', C respectively. It is known that among the obtained tetrahedra there exists one with the minimal volume. Express this minimal volume in terms of V.

Second Day

4. Find all pairs of real numbers (x,y) which satisfy the system

$$\begin{aligned} x-y &= 7; \\ \sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2} &= 7. \end{aligned}$$

5. Given two non-intersecting chords AB and CD of a circle k and a length $a < CD$. Determine a point X on k with the following property: If lines XA and XB intersect CD at points P and Q respectively, then $PQ=a$. Show how to construct all such points X and prove that the obtained points indeed have the desired property.

6A. Prove the following statement: If a polynomial $p(x)=x^3+Ax^2+Bx+C$ has three real roots at least two of which are distinct, then

$$A^2+B^2+18C>0.$$

6B. Each point of a plane is colored in one of three colors: red, black and blue. Prove that there exists a rectangle in this plane whose vertices all have the same color.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com