17-th Hellenic Mathematical Olympiad 2000

Athens, January 29, 2000

Juniors

1. Given three non-collinear points in the plane, find a line which is equally distant from each of the points. How many such lines are there?

2. A three-digit number $\overline{\alpha\beta\gamma}$ (in decimal representation) is such that

   (i) its hundreds digit is eqaul to the sum of the other two digits, and

   (ii) $\beta(\gamma+1) = 52 - 4\alpha$.

   Find all such numbers.

3. On a past Mathematical Olympiad the maximum possible score was 5. The average score of boys was 4, whereas the average score of girls was 3.25. The overall average score was 3.60. Find the total number of partitipants, knowing that it was in the range from 31 to 50.

4. Four pupils decided to buy some mathematical books so that

   (i) everybody buys exactly three different books, and

   (ii) every two of the pupils buy exactly one book in common.

   What are the greatest and smallest number of different books they can buy?

Seniors

1. Consider a rectangle $AB\Gamma\Delta$ with $AB = \alpha$ and $A\Delta = \beta$. The point E on $A\Delta$ is such that $AE/E\Delta = 1/2$ and O is the center of the rectangle. Let M be any point on the line OM interior to the rectangle. Find the necessary and sufficient condition on α and β that the four distances from M to lines $A\Delta, AB, \Delta\Gamma, B\Gamma$ in this order form an arithmetic progression.

2. Find all prime numbers p such that $1+p+p^2+p^3+p^4$ is a perfect square.

3. Find the maximum value of k such that

$$\frac{xy}{\sqrt{(x^2+y^2)(3x^2+y^2)}} \le \frac{1}{k}$$

   holds for all positive numbers x and y.

4. The subsets $A_1, A_2, \dots, A_{2000}$ of a finite set M satisfy $|A_i| > 2|M|/3$ for each $i = 1, 2, \dots, 2000$. Prove that there exists $m \in M$ which belongs to at least 1334 of the subsets A_i.