19-th Hellenic Mathematical Olympiad 2002

February 16, 2002

Juniors

1. In the exterior of an equilateral triangle $AB\Gamma$ of side α we construct an isosceles right-angled triangle $A\Gamma\Delta$ with $\angle \Gamma A \Delta = 90°$. The lines ΔA and ΓB meet at point E.

   (a) Find the angle $\Delta B \Gamma$.

   (b) Express the area of triangle $\Gamma\Delta E$ in terms of α.

   (c) Find the length of $B\Delta$.

2. In the Mathematical Competition of HMS (Hellenic Mathematical Society) take part boys and girls who are divided into two groups: *Juniors* (at most 15 years old) and *seniors*. The number of the boys taking part of this year competition is 55% of the number of all participants. The ratio of the number of junior boys to the number of senior boys is equal to the ratio of the number of juniors to the number of seniors. Find the ratio of the number of junior boys to the number of junior girls.

3. Determine all triples of positive integers (x, y, z) with $x \le y \le z$ satisfying
$$xy + yz + zx - xyz = 2.$$

4. Prove that $1 \cdot 2 \cdot 3 \cdots 2002 < \left(\frac{2003}{2}\right)^{2002}$.

Seniors

1. The real numbers α, β, γ with $\beta\gamma \ne 0$ satisfy $\frac{1-\gamma^2}{\beta\gamma} \ge 0$. Prove that
$$10(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma^3) \ge 2\alpha\beta + 5\alpha\gamma.$$

2. A student of the National Technical University was reading advanced mathematics last summer for 37 days according to the following rules:

   (i) He was reading at least one hour every day;

   (ii) He was reading an integer number of hours, but not more than 12, each day;

   (a) He had to read at most 60 hours in total.

   Prove that there were some successive days during which the student was reading exactly 13 hours in total.

3. In a triangle $AB\Gamma$ we have $\angle\Gamma > 10^\circ$ and $\angle B = \angle\Gamma + 10^\circ$. We consider point E on side AB such that $\angle A\Gamma E = 10^\circ$, and point Δ on side $A\Gamma$ such that $\angle\Delta BA = 15^\circ$. Let $Z \neq A$ be a point of intersection of the circumcircles of the triangles $AB\Delta$ and $AE\Gamma$. Prove that $\angle ZBA > Z\Gamma A$..

4. (a) Positive integers p, q, r, a satisfy $pq = ra^2$, where r is prime and p, q are relatively prime. Prove that one of the numbers p, q is a perfect square.

   (b) Examine if the exists a prime p such that $p(2^{p+1} - 1)$ is a perfect square.