15-th Hellenic Mathematical Olympiad 1998

January 31, 1998

Seniors

1. Prove that for any integer $n > 3$ there exist infinitely many non-constant arithmetic progressions of length $n-1$ whose terms are positive integers whose product is a perfect n-th power.

2. For a regular n-gon, let M be the set of the lengths of the segments joining its vertices. Show that the sum of the squares of the elements of M is greater than twice the area of the polygon.

3. Prove that for any non-zero real numbers a, b, c,

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

4. Let a function $g : \mathbb{N}_0 \to \mathbb{N}_0$ satisfy $g(0) = 0$ and $g(n) = n - g(g(n-1))$ for all $n \geq 1$. Prove that:

   (a) $g(k) \geq g(k-1)$ for any positive integer k.

   (b) There is no k such that $g(k-1) = g(k) = g(k+1)$.