12-th International Mathematical Olympiad

Budapest – Keszthely, Hungary, July 8–22, 1970

First Day – July 13

1. Given a point M on the side AB of the triangle ABC, let r_1 and r_2 be the radii of the inscribed circles of the triangles ACM and BCM respectively while ρ_1 and ρ_2 are the radii of the excircles of the triangles ACM and BCM at the sides AM and BM respectively. Let r and ρ denote the respective radii of the inscribed circle and the excircle at the side AB of the triangle ABC. Prove that $\frac{r_1}{\rho_1}\frac{r_2}{\rho_2}=\frac{r}{\rho}$. *(Poland)*

2. Let a and b be the bases of two number systems and let
$$A_n=\overline{x_1x_2\dots x_n}^{(a)},\qquad A_{n+1}=\overline{x_0x_1x_2\dots x_n}^{(a)},$$
$$B_n=\overline{x_1x_2\dots x_n}^{(b)},\qquad B_{n+1}=\overline{x_0x_1x_2\dots x_n}^{(b)},$$
be numbers in the number systems with respective bases a and b, so that $x_0,x_1,x_2,\dots,x_n$ denote digits in the number system with base a as well as in the number system with base b. Suppose that neither x_0 nor x_1 is zero. Prove that $a>b$ if and only if $\frac{A_n}{A_{n+1}}<\frac{B_n}{B_{n+1}}$. *(Romania)*

3. Let $1=a_0\le a_1\le a_2\le\cdots\le a_n\le\cdots$ be a sequence of real numbers. Consider the sequence $b_1,b_2,\dots$ defined by
$$b_n=\sum_{k=1}^{n}\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}\ .$$
   (a) Prove that for all natural numbers n, $0\le b_n<2$.

   (b) Given an arbitrary $0\le b<2$, prove that there is a sequence $a_0,a_1,\dots,a_n,\dots$ of the above type such that $b_n>b$ is true for an infinity of natural numbers n. *(Sweden)*

Second Day – July 14

4. For what natural numbers n can the product of some of the numbers $n,n+1,n+2,n+3,n+4,n+5$ be equal to the product of the remaining ones? *(Czechoslovakia)*

5. In the tetrahedron $ABCD$, the edges BD and CD are mutually perpendicular, and the projection of the vertex D to the plane ABC is the intersection of the altitudes of the triangle ABC. Prove that
$$(AB+BC+CA)^2\le 6(DA^2+DB^2+DC^2)\ .$$
For which tetrahedra does equality hold? *(Bulgaria)*

6. Given 100 points in the plane, no three of which are on the same line, consider all triangles that have all their vertices chosen from the 100 given points. Prove that at most 70% of these triangles are acute-angled. *(Soviet Union)*

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