19-th Indian Mathematical Olympiad 2004

1. In a convex quadrilateral $ABCD$, K,L,M,N are the midpoints of the sides AB,BC,CD,DA, respectively. Let BD bisect KM at Q. Suppose that $QA = QB = QC = QD$ and $\frac{LK}{LM} = \frac{CD}{CB}$. Prove that $ABCD$ is a square.

2. Given a prime number $p > 3$, find all papirs of integers (a,b) such that

$$a^2 + 3ab + 2p(a+b) + p^2 = 0.$$

3. If a is a real root of the equation $x^5 - x^3 + x - 2 = 0$, show that $[a^6] = 3$.

4. Let R be the circumradius of a triangle ABC, a,b,c be its sides, and r_a, r_b, r_c be the corresponding exradii. If $r_a \geq 2R$, prove that

   (a) $b < a$ and $c < a$;

   (b) $r_b < 2R$ and $r_c < 2R$.

5. Let S be the set of all hextuples (a,b,c,d,e,f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set $T = \{abcdef \mid (a,b,c,d,e,f) \in S\}$. Find the greatest common divisor of all the members of T.

6. Show that the number of 5-tuples (a,b,c,d,e) of positive integers satisfying

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

is odd.

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