# 21-st Indian Mathematical Olympiad 2006

1. The sides $a, b, c$ of a non-equilateral triangle $ABC$ form an arithmetic progression. Let $I$ and $O$ be the incenter and circumcenter of $\triangle ABC$ respectively.

   (a) Prove that $IO$ is perpendicular to $BI$.

   (b) If the lines $BI$ and $AC$ meet at $K$ and $D, E$ are the midpoints of $BC, BA$ respectively, prove that $I$ is the circumcenter of $\triangle DKE$.

2. Prove that for every positive integer $n$ there is a unique ordered pair $(a, b)$ of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2)+a.$$

3. For each triple $(a, b, c)$ of integers, define

$$f(a,b,c) = (a+b+C, ab+bc+ca, abc).$$

   Find all triples $(a,b,c) \in \mathbb{Z}^3$ such that $f(f(a,b,c)) = (a,b,c)$.

4. Some 46 unit squares of a $9 \times 9$ board are colored red. Show that there exists a $2 \times 2$ square containing at least three red unit squares.

5. In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120^\circ$, and $\angle ABD = 30^\circ$. Prove that

   (a) $c \geq a + b$;

   (b) $\left|\sqrt{c+a} - \sqrt{c+b}\right| = \sqrt{c-a-b}$.

6. (a) Prove that for each $n \geq 4011^2$ there is an integer $l$ with $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.

   (b) Find the smallest $M$ such that, for each integer $n \geq M$, there is an integer $l$ with $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.



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