23-rd Indian Mathematical Olympiad 2008

January 20

1. In a triangle ABC, I is the incenter, A_1, B_1, C_1 the reflections of I in BC, CA, AB respectively, and I_1 the incenter of triangle $A_1B_1C_1$. Suppose that the circumcircle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 lie on a circle.

2. Find all triples (p, x, y) of natural numbers with p prime such that $p^x = y^4 + 4$.

3. A set A of real numbers with at least four elements has the property that $a^2 + bc$ is a rational number for all distinct $a, b, c \in A$. Prove that there exists a positive integer M such that $a\sqrt{M}$ is rational for every $a \in A$.

4. All points with integer coordinates in the coordinate plane are colored using three colors, red, blue and green, each color being used at least once. It is known that point $(0,0)$ is red and $(0,1)$ is blue. Prove that there exist three points of different colors which form a right-angled triangle.

5. Let ABC be a triangle. Consider three equal disjoint circles τ_a, τ_b, τ_c inside $\triangle ABC$ such that τ_a touches AB and AC, τ_b touches BC and BA, and τ_c touches CA and CB. Let τ be the circle touching τ_a, τ_b and τ_c. Prove that the line joining the circumcenter O and the incenter I of $\triangle ABC$ passes through the center of τ.

6. Given any polynomial $P(x)$ with integer coefficients, show that there exist two polynomials $Q(x)$ and $R(x)$ with integer coefficients such that

   (i) $P(x)Q(x)$ is a polynomial in x^2, and

   (ii) $P(x)R(x)$ is a polynomial in x^3.