8-th Indian Mathematical Olympiad 1993

1. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at point P. Let O be the circumcenter of triangle APB and H be the orthocenter of triangle CPD. Show that the points H,P, and O lie on a line.

2. Consider a quadratic polynomial $P(x)=x^2+ax+b$ with $a,b\in\mathbb{Z}$. Show that for any integer n there is an integer m such that $P(n)P(n+1)=P(m)$.

3. If a,b,c,d are positive numbers with $a+b+c+d=1$, prove that
$$ab+bc+cd\le\frac{1}{4}\,.$$
Does the analogous inequality hold for n variables?

4. Find the set of all points P in the set of a triangle ABC such that $P\ne A,B,C$ and the triangles ABP, BCP, and CAP have the same circumradii.

5. Show that there exists a natural number n such that $n!$ in decimal system ends in exactly 1993 zeros.

6. Let $\mathscr{S}$ be the circumcircle of a right triangle ABC with $\angle A=90^\circ$. Circle $\mathscr{S}_1$ is tangent to the lines AB and AC and internally to $\mathscr{S}$. Circle $\mathscr{S}_2$ is tangent to AB and AC and externally to $\mathscr{S}$. If r_1 and r_2 are the radii of S_1 and S_2 prove that $r_1\cdot r_2$ equals four times the area of $\triangle ABC$.

7. Let B be a 53-element subset of $A=\{1,2,3,\dots,100\}$. Prove that there are two distinct elements $x,y\in B$ whose sum is divisible by 11.

8. Let f be a bijective function from $A=\{1,2,\dots,n\}$ to itself. Prove that there is a positive integer M such that $f^M(i)=i$ for each $i\in A$, where $f^M=f\circ f\circ\cdots\circ f$ (M times).

9. Prove that there exists a convex hexagon in the plane whose all interior angles are equal and whose side lengths are $1,2,3,4,5,6$ in some order.

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