Indian IMO Team Selection Tests 2001

First Test

1. Let x,y,z be positive numbers. Prove that if $xyz \geq xy+yz+zx$, then $xyz \geq 3(x+y+z)$.
2. Two symbols A and B obey the rule $ABBB = B$. Given a word $x_1x_2\dots x_{3n+1}$ consisting of n letters A and $2n+1$ letters B, show that there is a unique cyclic permutation of this word which reduces to B.
3. In a triangle ABC with incircle Γ and incenter I, the segments AI,BI,CI cut Γ at D,E,F, respectively. Rays AI,BI,CI meet the sides BC,CA,AB at L,M,N respectively. Prove that

$$AL+BM+CN \leq 3(AD+BE+CF).$$

When does equality occur?

Second Test

1. For any positive integer n, show that there exists a polynomial $P(x)$ of degree n with integer coefficients such that $P(0),P(1),\dots,P(n)$ are all distinct powers of 2.
2. Let $Q(x)$ be a cubic polynomial with integer coefficients. Suppose that a prime p divides $Q(x_j)$ for $j=1,2,3,4$, where $x_1,\dots,x_4$ are distinct integers from the set $\{0,1,\dots,p-1\}$. Prove that p divides all the coefficients of $Q(x)$.
3. Find the number of all unordered pairs $\{A,B\}$ of subsets of an 8-element set such that $A\cap B \neq \emptyset$ and $|A| \neq |B|$.

Third Test

1. Given a triangle ABC, triangles AEB and AFC are constructed externally such that $AE = EB$, $\angle AEB = 2\alpha$ and $AF = FC$, $\angle AFC = 2\beta$. Triangle BDC is constructed externally such that $\angle DBC = \beta$ and $\angle DCB = \alpha$.
   (a) Prove that DA is perpendicular to EF.
   (b) If T is the projection of D on BC, prove that $2\frac{DT}{BC} = \frac{DA}{EF}$.
2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(f(x)-x) = 2x$ for all $x > 0$.
3. Points $B = B_1,B_2,\dots,B_n,B_{n+1} = C$ are chosen on side BC of a triangle ABC in that order. Let r_j be the inradius of triangle AB_jB_{j+1} for $j = 1,\dots,n$, and r be the inradius of $\triangle ABC$. Show that there is a constant λ independent of n such that

$$(\lambda - r_1)(\lambda - r_2)\cdots(\lambda - r_n) = \lambda^{n-1}(\lambda - r).$$

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Fourth Test

1. Complex numbers α, β, γ have the property that $\alpha^k + \beta^k + \gamma^k$ is an integer for every natural number k. Prove that the polynomial $(x-\alpha)(x-\beta)(x-\gamma)$ has integer coefficients.

2. Let $p > 3$ be a prime. For each $k \in \{1, 2, \dots, p-1\}$, define x_k to be the unique integer in $\{1, \dots, p-1\}$ such that $kx_k \equiv 1 \pmod{p}$ and set $kx_k = 1 + pn_k$. Prove that
$$\sum_{k=1}^{p-1} kn_k \equiv \frac{p-1}{2} \pmod{p}.$$

3. Each vertex of an $m \times n$ grid is colored blue, green or red in such a way that all the boundary vertices are red. We say that a unit square of the grid is *properly colored* if

   (i) all the three colors occur at the vertices of the square, and

   (ii) one side of the square has the endpoints of the same color.

   Show that the number of properly colored squares is even.

Fifth Test

1. Let Γ be an arc of a circle passing through the vertices A and C of a rectangle $ABCD$. Circle Γ_1 is tangent to the sides AD and DC and Γ, and circle Γ_2 is tangent to the sides AB and BC and Γ, both Γ_1 and Γ_2 lying entirely inside the rectangle $ABCD$. Let r_j be the radius of Γ_j and r be the inradius of $\triangle ABC$.

   (a) Prove that $r_1 + r_2 = 2r$.

   (b) Show that one of the transverse common tangents to Γ_1 and Γ_2 is parallel to AC and has the length $|AB - BC|$.

2. A strictly increasing sequence (a_n) has the property that $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all $m, n \in \mathbb{N}$. Suppose k is the least positive integer for which there exist positive integers $r < k < s$ such that $a_k^2 = a_r a_s$. Prove that $r \mid k$ and $k \mid s$.

3. Let $P(x)$ be a polynomial of degree n with real coefficients and let $a \geq 3$. Prove that
$$\max_{0 \leq j \leq n+1} |a^j - P(j)| \geq 1.$$

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