15-th Iranian Mathematical Olympiad 1997/1998

Second Round

Time: 4 hours each day.

First Day

1. Suppose that a,b,x are positive integers such that
$$x^{a+b} = a^b b.$$
Prove that $a = x$ and $b = x^x$.

2. In an acute triangle ABC, points D,E,F are the feet of the altitudes from A,B,C, respectively. A line through D parallel to EF meets AC at Q and AB at R. Lines BC and EF intersect at P. Prove that the circumcircle of triangle PQR passes through the midpoint of BC.

3. Let $S = \{x_0, x_1, \dots, x_n\}$ be a finite set of numbers in the interval $[0,1]$ with $x_0 = 0$ and $x_1 = 1$. We consider pairwise distances between numbers in S. If every distance that appears, except the distance 1, occurs at least twice, prove that all the x_i are rational.

Second Day

4. Let ABC and XYZ be triangles and let
$$\begin{array}{lll} BC \cap ZX = A_1, & CA \cap XY = B_1, & AB \cap YZ = C_1, \\ BC \cap XY = A_2, & CA \cap YZ = B_2, & AB \cap ZX = C_2. \end{array}$$
Prove that
$$\frac{C_1C_2}{AB} = \frac{A_1A_2}{BC} = \frac{B_1B_2}{CA} \quad \text{if and only if} \quad \frac{A_1C_2}{XZ} = \frac{C_1B_2}{ZY} = \frac{B_1A_2}{YX}.$$

5. Let x,y,z be real numbers greater than 1 such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove that
$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}.$$

6. Let $\mathscr{P}$ be the set of all points in $\mathbb{R}^n$ with rational coordinates. For points $A,B \in \mathscr{P}$, one can move from A to B if the distance AB is 1. Prove that every point in $\mathscr{P}$ can be reached from any other point in $\mathscr{P}$ by a finite sequence of moves if and only if $n \geq 5$.

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