18-th Iranian Mathematical Olympiad 2000/01

Third Round

Time: 4 hours each day.

First Day

1. In an $n \times n$ matrix, a *generalized diagonal* is a set of n entries, one from each row and column. Let A be a $0,1$-matrix having exactly one generalized diagonal all of whose entries are 1. Prove that one can permute the rows and columns of A to obtain an upper triangular matrix.

2. Let O be the circumcenter and N the center of the nine-point circle of a triangle ABC. Let N' be the point such that $\angle N'BA = \angle NBC$ and $\angle N'AB = \angle NAC$. The perpendicular bisector of OA meets BC in A'. Points B' and C' are defined analogously. Prove that A', B', C' line on a line l that is perpendicular to ON'.

3. Let A be the set of sequences of integers $(x_1, x_2, \dots)$ and let $\phi : A \to \mathbb{Z}$ be a function such that:

   (i) $\phi(s+t) = \phi(s) + \phi(t)$ for all $s, t \in A$;

   (ii) $\phi(0, 0, \dots, 1, 0, 0, \dots) = 0$.

   (a) Prove that $\phi(1, 2, 4, 8, 16, \dots) = 0$.

   (b) Prove that $\phi \equiv 0$.

Second Day

4. Let $n = 2^m + 1$ and $f_1, \dots, f_n : [0,1] \to [0,1]$ be increasing functions that satisfy
$$|f_i(x) - f_i(y)| \le |x - y| \quad \text{for all } i \text{ and } x, y \in [0,1],$$
and $f_i(0) = 0$ for all i. Prove that there exist distinct i, j such that
$$|f_i(x) - f_j(x)| \le \frac{1}{m} \quad \text{for all } x \in [0,1].$$

5. In a triangle ABC, I and I_a denote the incenter and the excenter corresponding to side BC. Let A' and M respectively be the intersections of II_a with BC and the circumcircle of $\triangle ABC$, let N be the midpoint of arc MBA, and let S, T be the intersection points of rays NI and NI_a with the circumcircle of $\triangle ABC$. Prove that S, T, and A' are collinear.

6. By an n-variable formula we mean a function of n variables $x_1, \dots, x_n$ that can be expressed as a composition of $\max(a, b, \dots)$ and $\min(a, b, \dots)$. Suppose that P and Q are two n-variable formulas such that
$$P(x_1, \dots, x_n) = Q(x_1, \dots, x_n) \quad \text{for all } x_i \in \{0, 1\}.$$
Prove that $P \equiv Q$.

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