17-th Italian Mathematical Olympiad 2001

Cesenatico, May 4, 2001

1. A hexagon has all its angles equal, and the lengths of four consecutive sides are 5, 3, 6 and 7, respectively. Find the lengths of the remaining two edges.

2. In a basketball tournament every two teams play two matches. As usual, the winner of a match gets 2 points, the loser gets 0, and there are no draws. A single team wins the tournament with 26 points and exactly two teams share the last position with 20 points. How many teams participated in the tournament?

3. Consider the equation $x^{2001} = y^x$.

   (a) Find all solutions (x,y) with x prime and y a positive integer.

   (b) Find all solutions (x,y) in positive integers.

   (Recall that $2001 = 3 \cdot 23 \cdot 29$.)

4. A positive integer is called *monotone* if has at least two digits and all its digits are nonzero and appear in a strictly increasing or strictly decreasing order.

   (a) Compute the sum of all monotone five-digit numbers.

   (b) Find the number of final zeros in the least common multiple of all monotone numbers (with any number of digits).

5. The incircle γ of a triangle ABC touches AB at T. Let D be the point on γ diametrically opposite to T, and let S bt eh intersection of lines CD and AB. Show that $AT = SB$.

6. One hundred lamps are arranged in ten rows and ten columns. Some of them are on and the other are off. For each lamp there is a push-button that, when pressed, switches all lamps which are in the same row or column (including the lamp itself).

   (a) Find all states from which it is possible to light all the lamps.

   (b) Solve the same problem if there are 81 lamps in 9 rows and 9 columns.

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